SYMBOLIC LOGIC.

SYMBOLIC LOGIC

BY

JOHN VENN, Sc.D.; F.R.S.

FELLOW, AND LECTURER IN THE MORAL SCIENCES,
GONVILLE AND CAIUS COLLEGE, CAMBRIDGE.

SECOND EDITION. REVISED AND REWRITTEN.

"Sunt qui mathematicum vigorem extra ipsas scientias, quas vulgo mathematicas appellamus, locum habere non putant. Sed illi ignorant, idem esse mathematice scribere quod in forma, ut logici vocant, ratiocinari."

LEIBNITZ, *De vera methodo Philosophiæ et Theologiæ* (about 1690).

"Cave ne tibi imponant mathematici logici, qui splendidas suas figuras et algebraicos mæandros universale inventionis veri medium crepant."

RÜDIGER, *De sensu veri et falsi*, Lib. II. Cap. IV. § XI. (1722).

London:
MACMILLAN AND CO.
AND NEW YORK.
1894

Cambridge:
PRINTED BY C. J. CLAY, M.A. AND SONS,
AT THE UNIVERSITY PRESS.

TABLE OF CONTENTS.

PREFACE.

I HAVE so fully explained the nature and aim of this system of Logic, in the Introduction, that nothing further need be said on this head. The substance of most of these chapters has been given in my college lectures, our present intercollegiate scheme of lecturing (now in operation for about twelve years) offering great facilities for the prosecution of any special studies which happen to suit the taste and capacity of some particular lecturer and a selection of the students. I mention this in order to explain what might seem a disproportionate devotion of time to one peculiar development of Logic.

Besides Mr C. J. Monro, who has repeated the kind help he gave on a former occasion, I have to thank several friends and fellow-lecturers (amongst whom I must mention Mr H. Sidgwick, Mr H. S. Foxwell, Mr J. Ward, and Mr J. N. Keynes, before whom several of these chapters were read and discussed) for many valuable suggestions and criticisms. I must also express my obligations to Prof. G. Croom Robertson, of University College, London, for his kindness in procuring me several rare works from the valuable library of his college.

I should add here that the substance of some of the following chapters has already appeared elsewhere: viz. Chap. I in *Mind* (July, 1880), Chap. V in the *Philosophical Magazine* (July, 1880), and Chap. XX in the *Proc. of the Cambridge Philosophical Society* (Dec., 1880), but all of these have been rewritten and enlarged. The general view adopted in this work was sketched out in an article in the *Princeton Review* of September, 1880.

CAMBRIDGE,
April 7, 1881.

PREFACE TO THE SECOND EDITION.

At the time when the first edition of this work was composed it would scarcely be too much to say that the conception of a Symbolic Logic was either novel or repugnant to every professional logician. An amount of explanation and justification was therefore called for which does not seem to be quite so necessary now. I have endeavoured to meet the somewhat altered state of knowledge and feeling by such changes of exposition as could be conveniently introduced. The whole work has been carefully revised and rewritten, not a single page having been left unaltered. These alterations consist partly in condensation of explanation: partly in a fuller discussion of certain topics; such as particular propositions, and the intensive and propositional interpretation of our symbols. The chapter on Hypotheticals has been considerably modified, and the Historical notes in the final chapter much added to. Altogether there is a net addition of nearly a hundred pages. No fundamental alterations have been introduced; for, though I have now as a rule adopted the unexclusive notation for addition, I have always maintained that this was a question of method rather than of principle.

During the year expended on this edition I have endeavoured as far as possible to re-read the old works, and

to study carefully the new. Of these latter the most important seem to me the following:—Prof. Schröder's *Vorlesungen* (contains an admirably full and accurate discussion of the whole range of our subject, written prominently from the mathematician's point of view); The Johns Hopkins *Studies* (a collection of essays of much interest and originality); Dr Keynes' *Formal Logic* (written from the point of view of the formal logician, but proving that the most complicated problems can be solved with much less of symbolic apparatus than had previously been supposed); and Mr W. E. Johnson's *Essays in Mind*. To these may be added a great variety of original problems which have appeared from time to time in the *Educational Times*, with solutions by different logicians. These and the remaining works, so far as they have fallen under my notice, are referred to in the notes and in the Bibliographical list at the end. I should add that the problem refered to on p. 442 has been since discussed by its proposer, in the last number of *Mind*.

I owe much, in the way of criticism and suggestion, to the knowledge and acuteness of Mr W. E. Johnson, who has kindly looked through all the proof sheets for me.

GONVILLE AND CAIUS COLLEGE,
Oct. 1894.

INTRODUCTION.

(Slightly altered from the first edition.)

THERE is so certain to be some prejudice on the part of those logicians who may without offence be designated as anti-mathematical, against any work professing to be a work on Logic, in which free use is made of the symbols $+$ and $-$, $\times$ and $\div$ (I might almost say, in which x and y occur in place of the customary X and Y), that some words of preliminary explanation and justification seem called for. Such persons will probably without hesitation pronounce a work which makes large use of symbols of this description to be mathematical and not logical. Though such an objection betrays, as I shall hope to show, some ignorance as to the nature and functions of mathematics[1], yet there is certainly a fragment of truth suggested by it.

This truth consists in what may be described as being the present accidental dependence of the Symbolic Logic upon Mathematics. Those generalizations to which Boole was the first to direct attention, and which (as will be fully

[1] I may remark that by 'mathematics' here I refer, broadly speaking, to questions of number, magnitude, shape and position. I am quite aware that exponents of the higher branches would not admit the exhaustiveness of this description, but it will well include all that can be meant by those with whom I am arguing.

explained in due place) are purely logical in their origin and nature, are yet of a character which it is very difficult for any one to grasp who has not acquired some familiarity with mathematical formulæ. This should not be so, and I hope will not always continue to be so, but at present it is almost inevitable. The state of things arises from the fact that we have to realise a certain generalized use of symbols; that is, to take account of an extension of their use to cases distinct from, but analogous to, that with reference to which they first had their meanings assigned. In such extension, it must be understood, we have to anticipate not only a wider range of application, but also a considerable transfer of actual signification. This deliberate and intentional transfer of signification to analogous cases, as it happens, is perfectly familiar to the mathematician, but very little recognized or understood in other departments of thought. Hence it comes about that when people meet with such an expression as $x+y$ in a work which is professedly logical, they are apt to say to themselves, 'This is a mathematical symbol: the author is diverging into mathematics, or at least borrowing something from that science.' If they were to explain themselves they would probably maintain that the sign (+) stands for addition in mathematics, and therefore should not be resorted to when we have no idea of adding quantities together[1].

[1] Presumably this was Spalding's feeling when he insisted: "All attempts to incorporate into the universal theory of Thought a special and systematic development of relations of number and quantity must be protested against....No cumbrous scheme of exponential notation is needed, and none is sufficient, for the actual guidance of thought when its objects are not mathematical" (*Logic*, 1857, p. 50). He might have set his mind quite at rest as regards the relations of number and quantity. We have no more intention than himself of introducing these relations into our treatment of propositions:—in fact decidedly less so, inasmuch as

What such objectors must be understood to mean, or rather what they ought to mean, is of course that (+) is the sign of addition in *Arithmetic*, and of nothing else there; but they should not identify Arithmetic with the whole of mathematics. Even to go no further than Algebra, we find that (+) has already extended its signification, having come to include ordinary subtraction, in case the quantity to which it is prefixed has itself a negative value. Go a little further and we find that the same sign is used to indicate direction in space instead of merely ordinary addition. For instance, let any one take up some work imbued with the spirit of

we do not quantify predicates. 'Exponential notation' is, I suppose, here employed as a vague epithet of disapproval, but it is ill chosen; for whilst the rejection of all exponential notation is the formal differentia of the true logical calculus as compared with the mathematical, Mr Spalding's own work is almost the only one I have ever seen which *does* introduce exponents into Formal Logic. He writes A^2 and I^2 for 'All X is all Y', and 'Some X is some Y', giving as the merit of this notation that it "intimates a relation of the two forms to the received A and I; and the character of this relation is faintly hinted at when the added forms are symbolized as higher powers of the old ones"! (In saying this I must add that Spalding's treatise seems to me, on the whole, an excellent one.)

It is curious at what an early date the logicians began to get into this scare at the threatened inroad of mathematical notation, as the reader will see by the extract which I have given from Rüdiger on the title page. Any modern protester against Boole might quote his very words. But who these mathematical logicians are I have not yet succeeded in determining. The two editions of the work published in the author's lifetime appeared in 1709 and 1723. I know of nothing answering to such a description, so early as this, except a few letters and other short papers by Leibnitz (most of which cannot be said to have been published), and one by James Bernoulli. The only authors whom Rüdiger mentions in this context are, I think, Descartes, Spinoza and Tschirnhausen. The reference to the two former suggests that what he had in view was the employment of the so-called 'mathematical method,' viz. that in which we start with definitions, rather than an actual employment of mathematical notation. Prof. Adamson confirms me in this opinion.

modern analysis (Clerk Maxwell's Elementary treatise on *Matter and Motion* will answer the purpose), and he will perceive that $A + B$ has come to indicate a certain change both of magnitude and direction. Similarly with other successive transfers of signification. The full application of this principle to Logic must be reserved for discussion in future chapters. At present it will be enough to remark that before any one denounces what he considers to be a merely mathematical signification on the part of these symbols $(+, -, \times, \div)$ he should endeavour to ascertain with some degree of adequacy what range of interpretation they already possess within that science.

But it is here that the difficulty already alluded to makes itself felt, owing to which some acquaintance with mathematics becomes what I have called an accidental or practical necessity. We propose to carry out, in Symbolic Logic, an extension of the signification of our symbols similar to that just mentioned as existing in mathematics. Such an extension as this might conceivably, no doubt, be thoroughly attained within the province of Logic alone, for none but logical conceptions and processes will need to be appealed to. And when these developments of the subject are more generally known, and more fully worked out, this extended view possibly will be so attained. There is indeed no absolute reason, in the nature of things, why the logical calculus should not have been developed to its present extent before that of mathematics had made any start at all. The Greeks, for instance, had almost nothing which we should term symbolic language in use for their arithmetic or geometry, but there was no actual obstacle to their inventing for their own logical purposes all those symbols which Boole and others before him borrowed from the sister science. Had this been so there might in time have been corresponding

protests, on the part of the mathematicians, against the extension of these symbols to do duty in the new department, on the ground that they were an introduction of 'relations which are *not* of number or quantity.'

At the present time discussions of this particular kind,—that is, having reference to symbolic language in general, or to that of Logic and not Mathematics,—are hardly to be found anywhere. Boole's work of course implies much thought in this direction, but there is very little in the way of direct explanation offered by him; as is evidenced by the mere fact that he attempts no explanation whatever of his use of the sign of division in Logic. In fact the student must consult the works of some of the more philosophical mathematicians[1] in order to find what he wants; though naturally their discussions are so much confined to their own wants and aims, and often presuppose such an acquaintance with their own subject, that much of the value of what they say is lost to ordinary lay-readers.

This explanation of the principles of the logical calculus, in entire independence of those of the mathematical calculus, is one of the main objects which I have had before me in

[1] There is probably no better book in English, for this purpose, than De Morgan's *Trigonometry and Double Algebra*, which is however unfortunately out of print. Useful suggestions will also be found in the easier introductory parts of such a work on *Quaternions* as that of Kelland and Tait; and in the articles by De Morgan, in the Penny Cyclopædia, on *Negative quantities*. Of course the two former demand some familiarity with more than merely elementary mathematics.

There are several valuable works in German: *e.g.* Schröder's *Arithmetik und Algebra*, Hankel's *Vorlesungen über die Complexen Zahlen*, and Lipschitz's *Grundlagen der Analysis*. The first of these discusses very fully the nature of the elementary direct and inverse operations of Arithmetic, the others (I speak from a very slight acquaintance) are of too advanced a character to be readily intelligible to those who do not bring some corresponding knowledge with them.

writing this book. Every criticism and explanation offered here is to be regarded as standing on a purely logical basis, and therefore as being on all essential grounds well within the judgment and appreciation of any ordinary logician. I say 'essential grounds' in consequence of the accidental impediments above mentioned. It is at the present time almost impossible to find any good discussion of the nature of symbolic language in general except in the works of a few mathematicians; there can therefore be no disguising the fact that those who come here without some acquaintance with that science will stand at a certain disadvantage.

There can surely be no doubt as to the desirability of thus opening out a second means of acquiring an intelligent command of the principles of a really general symbolic language. Taking the most earnest anti-mathematician's own view as to the probable origin of the existent symbols, cannot they sympathise with the resolve of ~~Wesley~~ Luther—if he was the author of the saying—that the good tunes should not all remain in bad hands[1]? There is no vested right in the use of + and −, and it is therefore as open to them as to any one else to express themselves by aid of these symbols. So far as one can judge by the present course of the more abstract sciences, whether physical or mathematical, the value of symbolic language, in respect of the possibility of eliciting new and unexpected meanings out of old and familiar forms, is likely to be continually more appreciated. It will be a real gain to the cause of philosophy if some alternative branch of knowledge can be found which will

[1] There are reasons for supposing, according to De Morgan, that these signs + and − were not invented by the mathematicians, but borrowed by them from the practice of the counting house (*Camb. Phil. Tr.* No. XI. "On the early history of the signs + and − in mathematics").

afford a field for acquiring and realising such conceptions as that of this development of signification.

The relation of these languages of Logic and of Mathematics to one another as mere languages, signification apart; the extent to which they can be considered to be one and the same tongue; and the respects in which they differ from one another, may be familiarly illustrated as follows.

Suppose that any one came into a lecture room and saw the expression $A + B$ written on the board: Could he infer the subject of the lecture, or the meaning of the signs? Certainly not: but of one thing he may feel confident, viz. that, whichever of the various admissible meanings it may have borne, he will not do any mischief by transposing A and B, because the commutative law is tolerably certain to be accepted by any person who wrote down such an expression. Now let him be told that the lecture was one on Logic, and the language therefore in which the symbols were written was logical. He may go a step further, by putting some significance into the signs. He knows that A and B stand respectively for classes, and that $+$ aggregates those classes. But here again he comes to a stop. Though he knows the *language* of the lecturer he does not know his *dialect*, so to say. Was it the exclusive or the non-exclusive dialect for alternatives[1]? If the former, he can infer that A and B have nothing in common; as otherwise $A + B$ would have been bad grammar for 'A or B'; it ought then to have stood $A + \bar{A}B$. The dialect settled, he is of course still in entire darkness as to what A and B may mean. He cannot even know that they are classes strictly so called, for they may equally stand for single events or for propositions. For such detailed information, he must apply to the lecturer himself, or to one of his class; for nothing

[1] For explanation of the following remarks, see Chap. II. p. 46.

but specific information can possibly enable him to infer such facts as these.

But suppose on the other hand, that he were told that the lecture had been on mathematics, would his uncertainty be more completely and immediately removed? Certainly not. It is true that there are not here what I have just called dialectic varieties, because almost all who use the language of mathematics upon the same subject use it according to precisely the same laws; that is, the differences of usage depend here not upon the speaker but upon his subject. But the nature of that subject would make all the difference in the interpretation of the symbols. Had the lecture been on Arithmetic, he would know that A and B had stood generally for numbers, or specially for some number, and that + had stood for simple addition. Had it been on Statics, he would reasonably infer that A and B had stood for *forces*, and had indicated both their magnitude and their direction, and that + had stood for the *composition* of these forces, so that the whole expression represented the resultant force. But then again he would come to a stop. He could no more draw lines representing those forces, in the absence of information as to the scale and direction adopted by the lecturer, than he could say, without corresponding information from the logical lecturer, what sort of things *he* had in view with his A and his B.

We may therefore regard Symbolic Logic and Mathematics as being branches of one language of symbols, which possess some, though very few, laws of combination in common. This community of legislation or usage, so far as it exists, is our main justification for adopting one recognised system of symbols for both alike. The older branch,—that of mathematics,—may be divided, in reference to its mere form, into several different languages. The distinction be-

tween these depends almost entirely upon the nature of the subject matter. There is a language, for instance, of Algebra, and one of Quaternions, besides others which have been adopted or proposed. These differences are what may be called scientific, rather than dialectic. Of the latter there is not much trace in mathematics; in other words, there are few, if any, differences of symbolic usage depending not upon subject matter but upon the custom of individual writers or schools. The language of Logic on the other hand has hardly yet begun to show any subdivision according to the subject matter treated of, the present diversity of usage being one between different schools treating of one and the same subject. Where it shows signs of such differentiation is in respect of what is termed the Logic of Relatives. So far as this is cultivated we certainly demand some relaxation, if not entire rejection of, the law that $xx = x$, viz. that repetition of class attribution does not alter the result. Such a change as this is of course of a far more serious character, on any formal or systematic grounds, than the most extensive transfer of mere application of the rules of operation.

It ought to be pointed out here that some of those who maintain that this system of Logic is to be called 'mathematical,' labour under no confusion whatever as to the real nature of each of these subjects. With them it is mainly a question of definition, since they understand by mathematics any such language of pure symbols as I have above described, and which I prefer to regard as a genus which includes mathematics as one of its species. There can be little doubt that this was Boole's own view, for I think that Mr Harley is right (*Report of British Association*, 1866; see also the Report for 1870) in thinking that the term mathematical was used by him "in an enlarged sense,

as denoting the science of the laws and combinations of symbols, and in this view there is nothing unphilosophical in regarding Logic as a branch of Mathematics instead of regarding Mathematics as a branch of Logic[1]". The extract from Leibnitz which I have placed on the title page is couched in a similar spirit. There is nothing erroneous in this interpretation, provided we clearly understand that the principles and procedure of the logical calculus must be justified on their own account, in entire independence of any other science. When however we have, as here, two cognate branches of abstract science of almost equal antiquity, it seems to me to be likely to lead to misconception if we thus stretch the name of one of them so as to cover the other also. It is better to use some vaguer general name to include them both.

As regards the utility of the Symbolic Logic, the defence is sometimes rested almost entirely upon the great increase of power which it affords in the solution of complicated problems. I should be perfectly prepared to support its claims on this ground alone, even were there nothing else to be said in its favour. It is scarcely conceivable that any one who has been in the habit of using these symbolic methods should doubt that they enable us easily to thread our way through intricacies which would seriously tax, if they did not completely baffle, the resources of the syllogism. As well might one endeavour to discard the help of Algebra and persist in trying to work out our equations by the aid of Arithmetic only. But then it must be admitted that these

[1] Boole himself has said the same: "It is simply a fact that the ultimate laws of Logic,—those alone upon which it is possible to construct a science of Logic,—are mathematical in their form and expression; though not belonging to the mathematics of quantity." (*Lecture delivered at Cork*, 1851.)

really intricate problems are seldom forced upon us in any practical way. Regarded as means to an end, rather than as studies on their own account, Logic and Mathematics stand on a very different footing in this respect. We acquire skill with the weapons of the former rather with a view to our general culture, whereas in learning to use the latter we are also training ourselves for perfectly serious intellectual warfare. Without consummate mathematical skill, on the part of some investigators at any rate, all the higher physical problems would be sealed to us; and without competent skill on the part of the ordinary student no idea can be formed of the nature and cogency of the evidence on which the solutions rest. Mathematics are here not merely a gate through which we may approach if we please, but they are the only mode of approach to large and important districts of thought.

In Logic it is quite otherwise. It may almost be doubted whether any human being, providing he had received a good general education, was ever seriously baffled in any problem, either of conduct or of thought (examinations and the like of course excluded) by what could strictly be called a merely logical difficulty. It is not implied in saying this, that there are not plenty of fallacies abroad which the rules of Logic can detect and disperse, as well as abundance of fallacious principles and methods resorted to, which logical training can gradually counteract. The question is rather this:—Do we ever fail to get at a conclusion, when we have the data perfectly clearly before us, not from prejudice or oversight but from sheer inability to see our way through a train of logical reasoning? The mathematician is only too well acquainted with the state of things in which he has all the requisite data clearly before him (the problem being fully stated by means of equations), and yet he has

to admit that the present resources of his science are quite inadequate to effect a solution. It is almost needless to say that there is nothing resembling this in Logic; the difficulties which persistently baffle us here, when we really take pains to understand the point in dispute, being philosophical rather than logical. The collection of our data may be tedious, but the steps of inference from them are mostly very simple.

When it is said that the main advantages to be derived from the study of these extensions of Logic are speculative rather than practical, it may be well to spend a few pages in illustrating and expanding this statement. The general intellectual advantages of any serious mental exercise may, it is to be hoped, be taken for granted here. I will therefore start with the assumption that the ordinary Logic possesses some utility, and will only direct attention to such special advantages as the logical student may derive from our present subject, since these are more likely to be overlooked.

To begin then: the mere habit and capacity of *generalization* is surely worth a great deal. As one entire chapter is devoted to pointing out the various directions in which Symbolic Logic is to be regarded as generalizing the processes of the ordinary Logic, a few indications must suffice here. The common Syllogism may be described as being a solution of the following problem:—'Given the relations in respect of extension of each of two classes to a third, as conveyed by means of two propositions, find the relation of the two former to each other'. Of course it is not maintained that this is the only account to be given of the syllogist process, or even the most natural and fundamental one, b· it is certainly *one* account. Well, the general problem which this is a very special case, may be stated thus:— 'Given any number of propositions, of any kind, categoric

disjunctive or otherwise, and involving any number of terms, find the mutual relation to one another, in respect of their extension, of any selection from amongst all these terms to any other such selection'. Again, the syllogism is a case of *Elimination* as well as of inference; that is, we know that we thus get rid, in our conclusion, of one term out of the three involved in our premises. Here the corresponding general problem would be to ascertain whether there is any limit to the number of terms which can be thus eliminated from one proposition, or from any assigned group of propositions; and to give general rules for such a process of elimination.

A thorough generalization assumes sometimes an entirely unfamiliar aspect to those who were previously acquainted only with some very specialized form of the general process:—thus we all know what a difficult step it is to most beginners to extend 'weight' into 'universal gravitation.' In such cases the realization of the generalization may amount almost to the acquisition of a new conception, rather than to the mere extension of one with which we were already intimate. For instance, there is an inverse process, as distinguished from a direct, of which we may detect a rudimentary notion in what is called Accidental Conversion. Given that 'All X is Y', what is known about Y in relation to X? Logic, as we know, answers at once that 'Some Y is X', by which we may seem to mean, interpreting the statement in terms of extension, that the class Y certainly includes all X and for aught we know may include anything besides. Now differently worded, this process may be generalized as follows:—Given the relation of one class to another, find the relation of the second to the first[1]. The relation here is of course to be

[1] That is, Given $x=f(y)$, we express the inverse in the form $y=f^{-1}(x)$. This is, so far, a mere expression or

confined to those of which our Logic takes cognisance, but the classes themselves, it must be remembered, need not be of the simple *X* or *Y* type. On the contrary they may be composed of any number of terms, and these any how combined by aggregation, exception, restriction and so forth. When the process of Conversion is thus regarded, it is seen to be a limited case of a much more general problem.

The generalizations thus referred to are but one or two out of a number with which we shall have to occupy ourselves. What reason can be urged why those who are able to understand Logic thoroughly in its common form, should not also go on to study it in this extended form? Whatever objections we may feel to doing so let us not rest them on the ground that these discussions are no part of Logic. Say, if we like, that such questions are so simple that the traditional methods can readily grapple with them; say, if we like, that they are too intricate for profitable treatment; say, if we like, that they are ingenious trifles (we are all supposed to agree that the syllogism is eminently useful), but do not let us maintain that they are not logical. They belong to that science by a double right, both positively and negatively. Positively they belong to Logic because they are simply generalizations of processes of which Logic is universally admitted to take cognisance. Negatively they belong to it because they certainly do not belong to Mathematics, which is presumably the only other abstract science to which they

definition of what we want, and not at all a solution nor even a declaration that there is a solution. What is necessary is to rationalize the latter expression. The wide range which *f* assumes in mathematics precludes any general solution there. But, as will be found, there are such restrictions and simplifications upon all logical functions as to render a general solution perfectly feasible. It may be remarked, that the answer is an indefinite one, as is mostly case with inverse solutions; tha there is not one single answer, only one, to the problem.

are likely to be relegated. There is no more of enumeration or valuation of any kind of *unit* in them, than there is in the syllogism. They can be so stated as to involve nothing whatever but the mutual relations of various classes or class terms to each other in the way of inclusion and exclusion, of existence and of non-existence. So far from being fairly open to the charge of being too numerical, we are really more open to that of being almost prudishly averse to being supposed to enumerate at all, as will be found when we discuss the treatment of 'particular' propositions.

One great advantage of this kind of study is to be sought even within the sphere of ordinary Logic, in the increased clearness of view and philosophic comprehension which results from carrying our speculations somewhat outside that sphere. As De Morgan has said, "Every study of a generalization or extension gives additional power over the particular form by which the generalization is suggested. Nobody who has ever returned to quadratic equations after the study of equations of all degrees, or who has done the like, will deny my assertion that *οὐ βλέπει βλέπων* may be predicated of every one who studies a branch or a case, without afterwards making it part of a larger whole." (*Syllabus* p. 34.) So I should say here that the student will understand the nature of the simple logical processes in a better way when he has investigated the general processes of which they are particular cases. For instance, as I shall hope to show, one of the most essential characteristics of logical Elimination is loss of precision or determination; but this is a characteristic which is by no means obvious in the ordinary treatment.

To this should be added another important consideration. There are several rather perplexing questions in Logic in regard to which it does not seem possible to appreciate fully the grounds of decision, so long as we confine ourselves to

the narrow field afforded by the ordinary treatment. I may offer, as an instance in point, the explanation of the Import of Propositions proposed in Chap. VI. The view there propounded:—viz. that, for purely logical purposes, the only unconditional implication of even an Affirmative is to be found in what it *denies;* what it asserts being only accepted on the hypothesis that there are things existent corresponding to the subject and predicate:—will probably, on the first enunciation, sound very far-fetched and needless. In this systematic form the interpretation is, I believe, novel[1]; but a partial form of it, viz. the hypothetic interpretation of categoricals, has been frequently proposed and debated. Now I cannot but think that the very inadequate discussion of the subject, indeed its entire rejection from almost all English manuals, has arisen from the fact that the reasons for such a proposal cannot adequately be estimated within the boundaries of the common treatment. Within those boundaries the traditional explanation, or absence of one, will answer in a somewhat lame way; but when we seek for some account which shall be really general, adapted to any propositions or groups of propositions however complicated, we seem to be forced towards the explanation in question. The reader will find another analogous instance to this in our treatment of the word 'some', where the utter inadequacy of the scheme known as the Quantification of the Predicate will come out in the course of the discussion.

The solution of difficulties of the kind in question is often greatly aided by the careful examination of extreme or limiting values of our terms and operations. Popular feeling and popular thought, as a rule, object strongly to the intro-

[1] I think it was so when the above was first written. Practically all exponents of Symbolic Logic are in tolerable agreement now on this point.

duction of cases of this description, and not without reason on the whole. Far the larger part of the meaning of our statements is made up of implications, sometimes remote and subtle, and these rest upon various tacit restrictions as to the range of application of the terms in question. Hence our meaning would often be seriously disturbed by pushing on the interpretation of our terms to an extreme case[1]. But in any abstract science we are bound not to neglect the examination of such cases, since many valuable hints may be gained by their discussion; and under any circumstances every scientific statement is bound to be explicit and precise as to its limits. Hence I have made a point of going tolerably fully into enquiries of this nature wherever they have happened to lie near our path. I have been the more prompted to do so owing to the fact that most logicians have been far too ready to acquiesce in the popular prejudices on this point, instead of insisting that their own scientific language should be adapted to every scientific use.

Some of the remarks hitherto made will suggest the question whether it is proposed that the study of the Symbolic Logic should supersede that of the traditional Logic as a branch of education. By no means. No one can feel

[1] The mathematician of course is perfectly familiar with all this; but to the popular mind even extreme cases (*i.e.* such as fall within the given limits) would often seem to be nothing but attempts at a joke. Thus the statement that St Helena contains a large salt lake or sea, studded with islands, might call for some explanation as to what was meant. And yet, unless we insist upon some limit as to the relative extent of the land or water assignable to compose an island, what definition could be laid down for either lake or island, on a closed surface like a sphere, which should not make such a statement strictly true? Conceive one island on a globe of ocean, and let the former gradually increase till the relative proportions are inverted: at what stage are we to pronounce that the land includes the water rather than that the water includes the land?

more strongly than I do the merits of the latter as an educational discipline. And this conviction is even enhanced by the fact that some of the most instructive portions of the common system are just those which Symbolic Logic finds it necessary to pass by almost without notice. Amongst these may be placed the distinction between Denotation and Connotation, the doctrine of Definition, and the rules for the Conversion and Opposition of propositions as treated from the common point of view. Perfect clearness of apprehension on all these points seems essential to accuracy of thought, and it is difficult to find any better means of acquiring this clearness than the study of some of the ordinary logical manuals. Indeed one merit of the common system seems to me to lie in the comparative empiricism and restriction of its point of view; for it is this which enhances its educational value, by keeping its rules and forms of expression in tolerably close harmony with the language of ordinary life[1]. It is often, as we know, difficult to say what is a

[1] Some philologists have recently directed an attack against the whole science of Formal, viz. Aristotelian or Scholastic Logic, on the ground that it is so largely made up of merely grammatical necessities or conventions; nearly all its rules being more or less determined by characteristics peculiar to the Aryan languages. Thus Mr Sweet (*Trans. of the Philological Society*, 1876: quoted and endorsed by Mr Sayce) says that as a consequence of philological analysis "the conversion of propositions, the figures, and with them the whole fabric of Formal Logic fall to the ground". It is no business of mine here to defend the old-fashioned Logic, which has plenty of champions still, so I merely remark that I cannot think its case so desperate as this. It is quite true (so far as I can judge) that its dependence upon the accidents of speech is much closer than logicians have generally admitted. This would be a serious consideration if we maintained that the common schedule of propositions represents the way in which men necessarily assert and reason, instead of (as I have urged in the following chapter) merely one of several ways in which they may do so. For myself, I regard this close dependence upon popular speech as being rather a merit than otherwise for educational

grammatical and what a logical question, owing to the fact that the forms of proposition in the ordinary logic are just those of common life with the least degree of modification consistent with securing accuracy of meaning. Common Logic should in fact be no more regarded as superseded by the generalizations of the Symbolic System than is Euclid by those of Analytical Geometry. And the grounds for retaining in each case the more elementary study seem to be identical. The narrower system has its peculiar advantage, owing to the fact that, being by comparison more concrete, it is easier for a beginner to understand; that there is thus less danger of its failing to exercise the thinking faculty and merely leading to dexterity in the use of a formula; and that it is much more closely connected with the practical experiences and needs of ordinary life. The more general system, on the other hand, has vastly extended capacity, practises much more thoroughly the faculty of abstraction, and corrects and enlarges the scientific bases of the narrower system.

I think then that the Common Logic is best studied on the old lines, and that the Symbolic Logic should be regarded as a Development or Generalization of it. It is for this reason that I cannot regard the attempts made, in such very different directions, by Hamilton and Jevons, with any great satisfaction. The petty reforms represented by the Quantification of the Predicate and its consequences, seem to me to secure the advantages of neither system. We cut ourselves loose from the familiar forms of speech, and yet we do not secure in return any of the advantages of wide generalization.

purposes. The Symbolic Logic is as nearly free from all such accidents of speech as anything dealing with human thought well can be. The fact that this is so, constitutes one great reason why I should be very sorry (as above remarked) to see the common Logic superseded by a more scientific rival.

Jevons' individual reforms in the direction of our Logic seem to me to consist mainly in excising from Boole's procedure everything which he finds an "obscure form", "anomalous", "mysterious", or "dark and symbolic" (*Pure Logic*, pp. 74, 75, 86). This he has certainly done most effectually, the result being to my thinking that nearly everything which is most characteristic and attractive in the system is thrown away. Thus every fractional form disappears, so does the important indeterminate factor $\frac{0}{0}$, and all the general functional expressions such as $f(x)$ and its derivatives. That these are symbolic I freely admit; but that they are dark, mysterious, and obscure, in any other sense than can be predicated of all which is worth a serious effort in abstract speculation, or that they are anomalous in any sense whatever, I wholly deny. In particular, as will be presently pointed out, to reject fractional forms is not merely to maim our system[1] by omitting the necessary inverse process to that of multiplication, but actually to fall back behind the stage reached by more than one logician in the last century.

To those who already know something of the subject the following brief indication will convey a sufficient account of what may be supposed to be characteristic and original in the following work:—The thorough examination of the Symbolic Logic as a whole, that is, its relation to ordinary Logic and ordinary thought and language: the establishment and explanation of every general symbolic expression and rule on purely logical principles, instead of looking mainly to its formal justification; and the invention and employment of a scheme of diagrammatic notation which shall be in true harmony with our generalizations. As this work is intended to be an independent study of the subject from its founda-

[1] Most symbolic logicians, I must admit, will not quite agree with me here.

tions, and in no sense a mere commentary or criticism upon Boole, I have never discussed his particular opinions except where this seemed likely to throw light upon my own treatment of the subject.

Such anticipations of Boole's principles and results as I have succeeded in discovering have, as a rule, been mentioned in notes at the appropriate places. I have been the more careful to introduce these notes owing to the remarkable absence of any such references in most recent treatises on Symbolic Logic; on which subject indeed one may say, as Hamilton did on another branch of science, that "it seems to have been fated that every writer should either be ignorant of or should ignore his predecessors" (*Discussions*, p. 183)[1]. It would certainly seem that Boole had no suspicion that any one before himself had applied algebraic notation to Logic: (I am far from urging this as a reproach, for few men have had slighter opportunities for research than were open to him during the greater part of his life: moreover any anticipations which he could have found of his principal generalizations are indeed scanty and remote). Professor Jevons has roundly asserted (*Principles of Science*, Ed. I. 1874, p. 39) that "Boole is the only logician in modern times who has drawn attention to" the logical law[2] that $A = AA$.

[1] These remarks are not intended to apply to the works of Peirce and Schröder.

[2] Three perfectly explicit anticipations may be mentioned here, besides that of Leibnitz referred to in the next note. *First*, Segner; "Subjecti enim idea cum se ipsa composita novam ideam producere nequit; pariterque; Linea est extensa, curvum est extensum, ergo linea curva est extensa, ubi prædicatum cum se ipso compositum non mutatur" (*Specimen Logicæ*, p. 148). *Secondly*, Lambert, as quoted presently. *Thirdly*, Ploucquet, who in discussing an example formally stated as '*N* is *g*, *N* is *r*, *N* is *f*', goes on to say, "Nun wäre ungereimt zu setzen ......*NNNgrf*, sondern es muss nur so ausgedrückt werden, *Ngrf*". (*Sammlung*, &c. p. 254:—Some account of the notation will be found in Chap. XXI.) I have little doubt

Frege (*Begriffsschrift*, 1877) has no reference to any symbolic predecessor except a vague mention of Leibnitz. R. Grassmann's *Begriffslehre* (1872) has, I think, no reference whatever to any predecessor in this line. Delbœuf wrote the original of his *Logique Algorithmique* (in the *Revue Philosophique*, 1876), without having heard of Boole, as Mr McColl wrote his papers without having read that work.

It is true that since Leslie Ellis and Mr Harley called attention to some of the pregnant hints given by Leibnitz[1], he at least has been occasionally referred to. But I hope that this volume will convince the reader that there are some more serious and successful attempts at Symbolic Logic,

that any one better acquainted than myself with the Leibnitzian and Wolfian logicians could add more such notices.

[1] Ellis first called attention to Leibnitz, in this connection, in a note to Vol. I. (p. 281) of the edition of Bacon's works by him and Spedding (1858); but the reference there given to p. 130 of Erdmann's edition is clearly wrong. Mr Harley suggested (*Rep. of Brit. Ass.*, 1866) that Ellis probably meant to refer to p. 103, where Leibnitz represents the proposition 'All A is B' in a form equivalent to $A=AB$, for when $A=B$ we have $A=AA$. A more appropriate passage, I think, is to be found in the *Non inelegans specimen demonstrandi in abstractis* (Erdmann, p. 95), where Leibnitz says "si idem secum ipso sumatur nihil constituitur novum, seu $A+A \infty A$", where ∞ is a sign of identity. At first sight this looks like a statement of Prof. Jevons' 'Law of Simplicity', viz. $A+A=A$. But we must remember that Leibnitz represented by his symbols *attributes* rather than *classes*, (this becomes evident when he gives a concrete instance: Homo – rationalis ∞ brutum, where the subtraction is plainly intensive not extensive). 'Addition' of attributes, as explained below (p. 461), is almost equivalent to 'multiplication' of classes. Accordingly the formula in question becomes really equivalent to Boole's $AA=A$. Jevons has also pointed out (*Pr. of Sc. Ed.* II. 1876, Pref. p. 13), a passage at p. 98, which is in closer literal accord: "ut *b* est *aa*, vel *bb* est *a*,... sufficit enim dici *a* est *b*".

Students of Leibnitz are well aware that several of his short logical essays were first published by Erdmann in 1840. All the principal logical extracts are also very conveniently brought together in a little volume by Kvêt (*Leibnitzens Logik*, 1857).

which deserve notice at the hands of those who undertake to treat the subject. At least it would be well to consult some of these before laying down what has *not* been said or done by any who have gone before us. Of the labours of one of these writers, Lambert,—"der unvergleichliche Mann", as Kant himself termed him,—it will be best to give a very succinct account here, since the references afforded by occasional notes will not yield any adequate impression of the remarkable progress he had made. To my thinking, he and Boole stand quite supreme in this subject, in the way of originality; and if the latter had knowingly built upon the foundation laid by his predecessor, instead of beginning anew for himself, it would be hard to say which of the two had actually done the most.

Summarily stated, then, Lambert had got as far as this. He fully recognized that the four algebraic operations of addition, subtraction, multiplication, and division, have each an analogue in Logic; and that they may there be respectively termed aggregation, separation, determination, and abstraction, and be symbolized[1] by $+$, $-$, $\times$, $\div$. He also perceived the *inverse* nature of the second and fourth as compared with the first and third[2]; and no one could state more clearly that we must not confound the mathematical with the logical signification[3]. He enunciates with perfect clearness the principal logical laws, such as the commutative,

[1] One of the clearest statements to this effect is in the *Logische Abhandlungen*, I. 150: but it is too long to quote. (My readers will find several extracts of similar purport in various notes in Chh. II. and III.) The logical equivalents he actually uses are Zusammensetzung, Absonderung, Bestimmung, Abstraction.

[2] "Die Operationen + und − sind einander entgegengesetzt und sie leiden einerlei Verwechselungen wie in der Algeber" (*Log. Ab.* II. 62).

[3] "Wir haben die Beweise der Zeichnungsart kurz angezeigt, die Zeichen selbst aus der Algeber genommen, und nur ihre Bedeutung allgemeiner gemacht" (*ib.* I. 137).

the distributive, and the associative[1], and (under restrictions to be presently noticed) the special law[2] $AA = A$. He develops simple logical expressions precisely as Boole does[3], though without assigning any generalized formulæ for the purpose. He fully understood that the distinctive merit of such a system was to be found in its capacity of grappling with highly complicated terms and propositions; and he accordingly applies it to examples which however simple they may seem to a modern symbolist represent a very great advance beyond the syllogism[4]. Moreover, in this spirit of generalization, he proposed an ingenious system of notation, of a 1 and 0 description, for the 2^n combinations which may be yielded by the introduction of n class terms or attributes[5].

[1] "Da man in vielen Sprachen das Adjectivum vor- und nachsetzen kann, so ist es auch einerlei ob man nR oder Rn setzt" (*Log. Ab.* I. p. 150).

"Da es in der Zeichenkunst einerlei ist ob man $a+b$ oder $b+a$ setzt" (*ib.* I. p. 33).

"Will man aber setzen $(m+n)A$, so ist dieses $=mA+nA$. Es sei

$$m=n+p+q$$

und $\quad A=B+C+D+E,$

so hat man

$mA=(n+p+q)(B+C+D+E)$..."

[2] "Man kann zu einem Begriffe nicht Merkmale hinzusetzen die er schon hat... weil man sonst sagen könnte ein *eisernes Eisen*" (*ib.* II. 133). The reason why he did not admit this law universally was (as presently noticed) that he endeavoured to make his formulæ cover *relations* as well as common logical predications. This comes out clearly in the following passage: "Wenn der Begriff $=\alpha$ ist, $\alpha\gamma$ das Geschlecht, $\alpha\gamma^n$ ein höheres Geschlecht, $\alpha\delta$ der Unterschied, $\alpha\delta^n$ ein höherer Unterschied, $\alpha\gamma+\alpha\delta=\alpha$ die Erklärung, $(\alpha\gamma+\alpha\delta)^n$ oder $\alpha(\gamma+\delta)^n$ eine höhere Erklärung" i.e. α being a true logical class term $\alpha^n=\alpha$; but γ, being a relative term, γ^n does not $=\gamma$ (*ib.* p. 133).

[3] His formula is $a=ax+a|x$ (where $a|x$ means a not-x, viz. our $a\bar{x}$). He also has $x+y=2xy+x|y+y|x$; just as Boole develops the expression.

[4] Take, for instance, the following:—

$F::H=S::(P+G)::V::(A+C+Se)$

as expressive of "Die Glückseligkeit des Menschen besteht in der Empfindung des Besitzes und Genusses der Vollkommenheiten des innerlichen und äusserlichen Zustandes". The sign :: here denotes a relation (*ib.* I. 56).

[5] His scheme is this. Let 1 represent the presence, and 0 the ab-

Hypothetical propositions, of a certain simple class, he interpreted and represented precisely as we should[1]. Still more noteworthy is the fact that in one passage at least he recognized that the inverse process, marked by division, is an *indeterminate* one[2].

These are the main truths of this kind which Lambert had seized. Whatever the defects and limitations in their expression, they represent a very remarkable advance on any thing known to have been done before him. Where he mainly went astray was, I think, in the following respects. Though he realized very clearly that logical division is the inverse of multiplication, he failed to observe the indefinite character commonly assumed by inverse operations:—that is, he failed to observe it except in certain special cases, as just pointed out. He sometimes regarded the inverse as being merely the *putting back* a thing, so to say, where it was before[3], and accordingly omitted altogether that surplus

sence of any attribute. Then, if we keep the order in which the terms stand in our expression unaltered, 10101 and 10111 will take the place of what we might indicate by $x\bar{y}z\bar{w}v$ and $x\bar{y}zwv$. He then compares the extent to which various complex terms thus agree with each other or differ. He also employs the slightly more convenient notation of letters and their negation, thus: ABC, $AB0$, $A00$, and so on, to stand for our ABC, $AB\bar{C}$, $A\bar{B}\bar{C}$ (*Log. Ab.* II. 134). Of course there are great imperfections in such a scheme.

[1] "Die allgemeinste Formel der hypothetischen Sätze ist diese, Wenn A ein B ist, so ist es C. Diese Formel kann allezeit mit der folgenden verwechselt werden; Alles A so B ist, ist C. Nun ist, Alles A so B ist $=AB$. Folglich, Alles AB ist C. Daher die Zeichnung $AB>C$ oder $AB=mC$" (*ib.* I. 128).

[2] "Wenn $x\gamma=a\gamma$, so ist

$$x=a\gamma\gamma^{-1}=a\frac{\gamma}{\gamma}.$$

Aber deswegen nicht allezeit $x=a$; sondern nur in einem einzigen Falle, weil x und a zwei verschiedene Arten von dem Geschlecht $x\gamma$ oder $a\gamma$ sein können. Wenn aber $x\gamma=a\gamma$ nicht weiter bestimmt wird, so kann man unter andern auch $x=a$ setzen" (*ib.* I. 9:—(as this expressly refers to relative terms only it is not at variance with the note at p. 84).

[3] "Auch ist klar dass man sich dabei Operationen muss gedenken können, wodurch die veränderte Sache

indefinite term yielded by logical division, which is so characteristic of Boole's treatment. Probably no logician before Boole (with the very doubtful exception of H. Grassmann, as mentioned in the note on p. 268) ever conceived a hint of this. As a consequence Lambert too freely uses mathematical rules which are not justifiable in Logic. For instance, from $AB = CD$ he assumes that we may conclude $A : C = D : B$.

Another point that misled Lambert was the belief that his rules and definitions would cover the case of *relative* terms[1]. This will explain what might otherwise seem a complete misapprehension of the very first principles of the Symbols of Logic, viz. his occasional admission of *powers*, e.g. of the difference between mA and m^2A. I think it is a mistake to endeavour thus to introduce relative terms, but, if we do so, we must clearly reject the law that $x^2 = x$, in the case of such terms.

In thus realizing what Lambert had achieved (I have purposely brought a number of extracts together as the only way of conveying a just idea of their combined effect, though several of them have been quoted elsewhere in this volume) the reader must remember that he by no means stood alone. Two of his friends and correspondents,—Ploucquet and J. G. Holland,—are worthy coadjutors; and such logical writings as they have left behind are full of interesting suggestions of a similar kind, as the reader will see by referring to their names under the references in the bibliographical index at the end. These men all took their

in den vorigen Stand könnte hergestellt werden. Diese Wiederherstellung giebt demnach den Begriff der reciproken Operationen, dergleichen im Kalkul + und −, × und ÷" (*Log. Ab.* II. 150).

[1] "Unter den Begriffen $M = A : B$ kommen einige vor, die sehr allgemein sind. Dahin rechnen wir die Begriffe; Ursache, Wirkung, Mittel, Absicht, Grund, Art und Gattung" (*Architectonic*, I. 82).

impulse from Leibnitz and Wolf. During the 80 or 90 years which elapsed from their day to that of Boole there is almost a blank in the history of the subject, for we cannot put the efforts of Maimon and Darjes into the same category, ingenious as these were. One cannot but speculate upon the causes of this total disregard of these remarkable speculations[1]; a disregard which had already astonished J. Bernoulli, the editor of some of Lambert's posthumous works and of his letters, and which has been so complete since then that I have never (till quite lately) even seen these speculations of his referred to by any modern symbolic logician. For myself I confess to an uneasy suspicion that, great as may have been the influence for good of Kant in philosophy, he had a disastrous effect on logical method. In any case it is instructive to notice the vigour and originality with which the science was being treated whilst the great philosopher was still to be spoken of as "Herr Immanuel Kant, Professor der Philosophie zu Königsberg in Preussen" (Lambert's *Briefwechsel*), with the monotonous

[1] Lambert's *Neues Organon* is frequently referred to, in connexion with his doctrine of the different principles which govern the four Syllogistic Figures; but his best Symbolic speculations are not to be found there. These are given most fully in the *Logische Abhandlungen;* but several of the important principles are also to be found in the *Architectonic;* in his *Briefwechsel* (Vol. I.); in his correspondence with Ploucquet, in the collected logical works of the latter; and in a paper in the *Nova Acta Eruditorum* for 1765. The most convenient work by Ploucquet which I have seen is the one just mentioned, and referred to throughout this volume as the "*Sammlung*". It contains the most distinctive of his logical treatises; also several reviews of these, and some interesting letters by Lambert and Holland. Several other works by him are noticed in the list at the end of this volume. I know of no independent logical treatise by Holland except a small volume entitled "*Abhandlung über die Mathematik, die allgemeine Zeichenkunst und die Verschiedenheit der Rechnungsarten*". (1764). There are however many letters by him in Lambert's *Briefwechsel.*

flood of logical treatises which spread over Germany for so long afterwards, and the wash of which reached us in the works of Hamilton and Mansel. I deeply admire the learning and acuteness of many of these works produced during the days of strictest preservation from mathematical encroachment, but confess that they seem to me rather narrow in comparison with what was produced when the spirit and the procedure of the sister science were more freely welcomed by the logician.

For the convenience of the reader a few brief bibliographical notes are added in conclusion. Boole's logical publications, so far as I know, are the following:—

The Mathematical Analysis of Logic (1847).

The Calculus of Logic (*Camb. and Dub. Math. Journal*, 1848).

The Claims of Science, a Lecture delivered at Cork (1851—touches slightly on the subject).

An Investigation of the Laws of Thought (1854).

Of Propositions numerically definite (*Camb. Phil. Tr.* XI.).

There does not seem to me to be anything much of value in the first three beyond what is given more fully in his mature work. There are also a number of papers by him, of a more mathematical kind, mostly on Probability, in the *Philosophical Magazine*, and elsewhere.

As regards the incidents of his life, there is a short account in an article by Mr Harley in the *British Quarterly Review* for July 1866. The reader will also be interested by three papers of a more domestic character in the *University Magazine* for Jan. Feb. and March, 1878. They are entitled "The Home side of a scientific mind"; and were written by his widow.

Though Boole's productions did not encounter the neglect

which befell those of Lambert[1], his admirers will most likely agree that it was far too long before they became appreciated and utilized as they deserve, even by those whose previous training might have been expected to attract them towards such speculations. They will agree also that those who have treated Logic from the traditional standpoint may still find much to learn from a study of such wide generalizations. I do not propose (as already remarked) that our methods should be incorporated into the common system, still less that they should supersede it; but one might well have expected some more serious attempts at criticism and exposition of their general spirit, purport, and place in the science of inference. There have been, it need not be said, many minor criticisms and references in Journals and inde-

[1] There is a curious correspondence in the circumstances of the two men. Each was born in very humble circumstances; was almost entirely self-educated,—that is, was trained at no University or superior school;—and had to do much of the work of his earlier life against the disadvantages of a pressure of routine and elementary educational work, and comparative absence of intercourse with scientific society. They were both first-rate mathematicians. They died at the same age, viz. forty-nine. Lambert was born at Mülhausen in Elsass,—the town was at that time connected with the Swiss Confederation,—in 1728. He lived for some years as tutor to the family of Count de Salis in Chur. He moved to Augsburg in the year 1759, where he was "agrégé à l'Académie électorale de Bavière, avec le titre de professeur honoraire, un traitement"... (*Biographie universelle*), and to Berlin in 1763, where he was "académicien pensionnaire." He died there in 1777. He had wide speculative interests, and his name must be added to the list of prophets who foretold the Revolution. "Die Sachen nicht Helvetien allein, sondern für ganz Europa sich zu einer bevorstehenden grossen Revolution anschicken, die aber freilich zum Ausbruche nicht ganz reif ist" (1770:—Letters II. 49). Some account of his life and works will be found in a short centenary memorial volume published by D. Huber at Basle in 1829 (J. H. Lambert nach seinem Leben und Wirken ... dargestellt). Also in the *Darstellung seiner kosmologischen und philosophischen Leistungen*, by J. Lepsius (1881).

pendent works, especially of recent years. All of these which I have seen, and which appeared deserving of notice, will be found referred to in the Index. Three notices of a more distinctly expository kind have been given respectively by Mr Harley, Prof. Bain, and Prof. Liard of Bordeaux. The two former (*Brit. Quarterly*, July, 1866; *Deductive Logic*, pp. 190—207) are very brief. The latter is of a more ambitious kind, being an attempt to give a general account of the "Modern English Logic," i.e. of the works of G. Bentham, Hamilton, De Morgan, Boole, and Jevons. Prof. Liard has evidently taken pains to study these authors, and the volume possesses the national merit of lively and transparently clear exposition of all that is understood, but its critical value seems to me of a humble order. The portions treating respectively of the works of Boole and Jevons had already appeared in substance in the *Revue Philosophique* for March and Sep., 1877.

SYMBOLIC LOGIC.

CHAPTER I.

ON THE FORMS OF LOGICAL PROPOSITION.

It has been mentioned in the Introduction that the System of Logic which this work is intended to expound contains not merely an extension of the ordinary methods—though this is perhaps its principal characteristic—but that it also involves a considerable departure from the ordinary point of view. This latter characteristic is one which has not always been sufficiently attended to in discussions upon the subject, and the neglect of it has blunted the point of much of the criticism on each side. It will be well therefore, before explaining the foundations on which the Symbolic Logic must be understood to rest, to give a brief discussion of the corresponding substructure in the case of some other systems with which the reader is likely to be more or less familiar. It will be readily understood that we shall not dig down any deeper than is absolutely necessary, and this is fortunately not far below the surface. Psychological questions need not concern us here; and still less those which are Metaphysical. Differences of this kind have very little, if any, direct relation to one logical method rather than another. All in

fact that we have to do is to examine, from the point of view of experience and common sense, what interpretation is took put upon the general nature of a proposition.

If we were constructing a complete Theory of Logic we should have to attack the question as to what is the *true* account, by which we should understand the most fundamental account, of the nature and import of a proposition, and on this point different accounts would be to some extent in direct hostility to one another. But when we are discussing methods rather than theories this is not necessarily so. The question then becomes, which is the most convenient account rather than which is the most fundamental; and convenience is dependent upon circumstances, varying according to the particular purpose we have in view. So far as we are now concerned there seem to be three different accounts of the import of a proposition; the ordinary or *predication* view, the *class inclusion and exclusion* view, and that which may be called the *compartmental* view. It may perhaps be maintained that one of these views must be more fundamental than the others, or possess a better psychological warrant, but it cannot be denied that they are all three tenable views; that is, that we may, if we please, interpret a proposition in accordance with any one of the three. And this is sufficient for our present purpose.

The question to be here discussed is simply this. What are the prominent characteristics of each of these distinct, but not hostile, views? What are their relative advantages and disadvantages; to what arrangement and division of propositional forms do they respectively lead; and, in consequence, which of them must be adopted if we wish to carry out the design of securing the widest extension possible of our logical processes by the aid of symbols?

The neglect of some such preliminary enquiry seems to ave led to error and confusion. Logicians have been too nuch in the habit of considering that there could be only ne account given of the import of propositions. In conequence, instead of discussing the number of forms of proosition demanded by one or the other view, they have ttempted to decide absolutely *the* number of forms. This as led, as every one acquainted with the subject is aware, a most bewildering variety of treatment in many recent gical works. And the minor, but very useful question, as o the fittest view for this or that purpose, has been lost from ight in the too summary decision that one view was right, nd consequently the others wrong.

Let us first look at the traditional four forms, *A*, *E*, *I*, *O*; n reference to which a very few words will here suffice. The light in which a proposition has to be interpreted on this view is primarily that of *predication*. We distinguish between subject and attribute here, and we assert that a given subject does or does not possess certain attributes. These traditional forms appear to be naturally determined by the ordinary needs of mankind, and the ordinary pre-logical modes of expressing those needs; all that Logic has done being to make them somewhat more precise in their signification than they conventionally are. They adopt, as just remarked, the natural and simple method of asserting or denying attributes of a subject, that is, of the whole or part of a subject; whence they naturally yield four forms,—the universal and particular, affirmative and negative. For all ordinary purposes they answer admirably as they are, and by a little management they can be made to comprise nearly all the simple forms of assertion or denial which the human nind can well want to express.

With regard to these forms it must be very decidedly

subject possess the assigned attribute. No doubt they sometimes decide this point indirectly. Thus, in the case of a universal negative proposition we can easily see that any thing which possesses the attributes in the predicate cannot possess the attributes distinctive of the subject; that is, that the proposition can be simply converted. But this does not seem to be any part of the primary meaning of the proposition, which thinks of nothing but asserting or denying an attribute, and does not directly enquire about the extent of that attribute, or where else it is or is not to be found.

As just remarked, these forms of proposition certainly seem to represent the most primitive and natural modes in which thought begins to express itself with accuracy[1]. By combining two or more of them together they can readily be made equivalent to much more complicated forms. Thus, by combining 'All X is Y' with 'All Y is X,' we obtain the expression 'All X is all Y,' or 'X and Y are coextensive,' and so forth. As these familiar old forms have many centuries of possession in their favour, and the various technical terms and rules for Conversion and Opposition, and for the Syllogism, have been devised for them, there seem to be very strong reasons for not disturbing them from the position they have so long occupied. At least this should only be done if it could be shewn either that they are actually insufficient to express what we require to express, or that they rest upon a wrong interpretation of the import of a proposition. The former is clearly not the case, for as was just remarked (and as no one would deny) a combination of two or more of these forms will express almost anything in the way of a non-

[1] At least this seems so in all the languages with which we need consider ourselves concerned. What might be the most natural arrangement of the forms of propositions in non-inflectional languages must be left to philologists to determine.

it could be shewn either that they are actually insufficient to express what we require to express, or that they rest upon a wrong interpretation of the import of a proposition. The former is clearly not the case, for as was just remarked (and as no one would deny) a combination of two or more of these forms will express almost anything in the way of a non-numerical statement. And as regards the latter, the point of this chapter is to shew that we are not necessarily tied down to one exclusive view as to the import of a proposition; a point which must be left to justify itself in the sequel. I should say, therefore, that whatever other view we may find it convenient to adopt for special purposes, such as that of sensible illustration or of solving intricate combinations of statements, there is no valid reason for not retaining the old forms as well. They may not be the most suitable materials for very complicated reasonings, but for the expression and improvement of ordinary thought and speech they are not likely to be surpassed.

So much for this view. Now suppose that, instead of regarding the proposition as made up of a subject determined by a predicate, we regard it as assigning the relations, in the way of mutual inclusion and exclusion, of two classes to each other. It will hardly be disputed that most propositions *can* be so interpreted. Of course, as already remarked, this interpretation may not be the most fundamental in a Psychological sense; but when, as here, we are concerned with logical methods merely, this does not matter. For the justification of a method it is clearly not necessary that it should spring directly from an ultimate analysis of the phenomena; it is sufficient that the analysis should be a correct one.

Now how many possible relations are there, in this respect of mutual inclusion and exclusion, of two classes to

each other? Clearly only five. For the question here, as I apprehend it, is this:—Given one class as known and determined in respect of its extent, in how many various relations can another class also known and similarly determined, stand towards the first? Only in the following: It can coincide with the former, can include it, be included by it, partially include and partially exclude it, or entirely exclude it. In every recognized sense of the term these are distinct relations, and they seem to be the only such distinct relations which can possibly exist[1]. These five possible

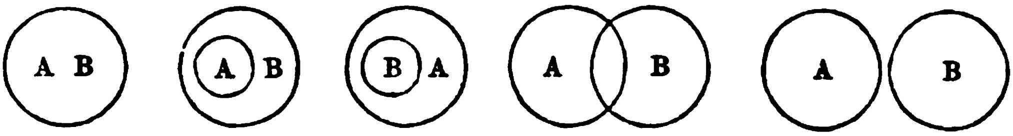
[1] That these are the only really distinct forms of objective class relation has been repeatedly recognized by logical writers. The earliest explicit statement of the matter that I have seen is by Gergonne (*Annales de Mathématiques*, VII. 189). He has gone very fully into the comparison between the five diagrammatical forms, and the four common propositional forms, ascertaining in detail (with aid of a peculiar system of notation) how those of the one set can be considered to correspond, collectively or individually, to any single form of the other set. He also notices the divergence of this scheme from the common one: "Il n'est aucune langue connue dans laquelle une proposition exprime précisément et exclusivement dans lequel de nos cinq cas se trouvent les deux termes qui la composent. Une telle langue si elle existait serait bien plus précise que les nôtres: elle aurait cinq sortes de propositions, et sa dialectique serait toute différente de celle de nos langues." F. A. Lange again (*Logische Studien*) has worked out the same view, emphasizing, even more strongly than I have done, the radically distinct theories as to the import of propositions involved in these two ways of stating or representing them. Other writers, for instance Twesten (*Logik*, p. 30) and the author of a little volume entitled *Thoughts on Logic, or the S. N. I. X. Propositional Theory*, reduce the five forms to four, by not distinguishing between (2) and (3); in other words by supposing that we have no means of distinguishing and recognizing one of the two classes from the other.

The relations of these five forms of proposition to each other, and to other schemes, as also the consequent schedules of possible forms of inference, have been very elaborately worked out by Schröder in the second volume of his *Vorlesungen*.

(The English reader will be interested in the fact that J. S. Mill, when a boy at Montpellier, attended a course of lectures on Logic by Gergonne. See his *Autobiography*, p. 58.)

arrangements would be represented diagrammatically as follows :—

1. 2. 3. 4. 5.



Before comparing in detail the verbal statement of these five forms with that of the four old ones, it must be pointed out how entirely the distinction between subject and predicate is robbed of its significance on such a scheme as this. The terms of the proposition here stand for two classes of things possessing some mutual relation of extension, so that the distinction between subject and predicate sinks into one which is purely grammatical. It is the merest accident which of the two classes occupies the first place in our verbal statement; whether, for instance, in (2) we say that *B* lies partly outside *A* or that *A* lies entirely inside *B*. Certainly when the diagrammatic representation alone was shewn to us no one could give a guess as to which circle was intended to stand for a subject and which for a predicate. We could not, that is, read the diagram off in one way and one way only, with confidence.

A very little consideration will serve to convince us that this scheme of five forms, and the old scheme of four, will not by any means fit in accurately with each other. Considering that they spring from different interpretations of the import of a proposition it could not be expected that they should do so. No. (5) is the one unambiguous exception, corresponding precisely to the universal negative 'No *A* is *B*.' That is, given 'No *A* is *B*,' we could only select this diagram; and conversely, given this diagram we could only describe it as 'No *A* is *B*' or 'No *B* is *A*.' But such correspondence

does not exist in any other case. Given 'All *A* is *B*' we could not but hesitate between diagrams (1) and (2); and if diagram No. (4) were chosen we should not know whether to describe it as 'Some *A* is *B*' or 'Some *A* is not *B*' or 'Some *B* is not *A*,' for it would fit either equally well.

Is there then any precise and unambiguous way of describing these five forms in ordinary speech? There is such a way, and to carry it out demands almost no violence to the usages of ordinary language. It is merely necessary to employ the word 'some,' and to say definitely that it shall signify 'some, not all'; a signification which on the whole seems more in accordance with popular usage than to say with most logicians that it signifies 'some, it may be all.' If we adopt this definition of the word our five diagrams will be completely, accurately, and unambiguously expressed by the five following verbal statements:—

All *A* is all *B*,
All *A* is some *B*,
Some *A* is all *B*,
Some *A* is some *B*,
No *A* is any *B*.

That is: given one of these statements, only one diagram out of this schedule could be selected for it: and conversely, given any one of these diagrams it could be matched with one only of these forms of words.

The tabular expression of these five forms will naturally recall to the reader's mind the well-known eight forms adopted by Hamilton[1] (*Logic* II. 277), viz.:—

[1] Hamilton's name is deservedly the best known in connection with this scheme, for the claim put forward in favour of Mr G. Bentham on the ground that he had (*Logic*, 1827) drawn up the same eightfold arrangement, seems to me quite untenable. For one thing, he had been anticipated by more than sixty years by Lambert, who in 1765 drew up a pre-

All *A* is all *B*.	Any *A* is not any *B*,
All *A* is some *B*,	Any *A* is not some *B*,
Some *A* is all *B*,	Some *A* is not any *B*,
Some *A* is some *B*,	Some *A* is not some *B*.

I might have termed the view as to the import of propositions now under discussion the Hamiltonian, instead of the class inclusion and exclusion view; and should have done so but for the fact that it is clearer and simpler to describe a system itself as one understands it, rather than to begin by giving what one cannot but regard as an erroneous expression

cisely similar table to that which is now so familiar to us (*Sammlung der Schriften welche den Logischen Calcul Herrn Prof. Ploucquet's betreffen*, p. 212). But in philosophical matters priority of mere statement is surely of but little value; appeal should rather be directed to the use made of a principle and to the evidence of its having been clearly grasped. Taking this test, the merit, such as it is, of the Quantification of the Predicate, must, I should think, be assigned to Ploucquet, and that of the closely connected doctrine of these eight propositional forms to Hamilton. As regards the Quantification, Ploucquet freely uses the distinctively characteristic forms 'No *A* is some *B*,' 'Some *A* is not some *B*;' and even distinguishes and directs attention to the case in which from two propositions of the form 'All *A* is some *B*,' 'All *C* is some *B*,' we can conclude that 'all *A* is all *C*;' viz. when the 'some *B*' is the *same* some (*Methodus Calculandi*). Nearly all the other consequences of the doctrine,—e.g. the simple conversion of the particular negative—are pointed out. He nowhere recognizes the appropriate table of the consequent eight propositions, though he was the means of suggesting it to Lambert. As regards mere priority of statement, it may be remarked that another writer, Mr Solly (*Syllabus of Logic*, 1839), had also given a similar table, before the time at which Hamilton, by his own assignment of dates, had begun to publicly teach this doctrine. But neither Mr Bentham nor Mr Solly seems to me to have understood exactly the sense in which their scheme was to be interpreted, nor to have attached much importance to it. (The reader interested in this subject will of course be acquainted with the controversy begun originally between Hamilton and de Morgan in 1846; continued intermittently in the *Athenæum*; and concluded in the *Contemporary Review* in 1873. There are several historic references also in Hamilton's *Logic*, II. 298, and Prof. Baynes's *New Analytic*).

of that system by some one else. Moreover there are obvious reasons for wishing to keep as free as possible from a somewhat prolonged and intricate controversy. At the same time, I must state my own very decided opinion that the view in question is that which Hamilton, and those who have more or less closely followed him in his tabular scheme of propositions, must be considered unconsciously to adopt.

The logicians in question do not seem to me, indeed, to have at all adequately realized the importance of the innovation which they were thus engaged in introducing; nor, it must be added, the utter inadequacy of the means they were adopting for carrying it out. What they were really at work upon was not merely the rearrangement, or further subdivision, of old forms of proposition, but the introduction of another way of looking at and interpreting the function of propositions. The moment we insist upon 'quantifying' our predicate we have to interpret our propositions in respect of their extension, that is, to regard them as expressing something about the actual mutual relations of two classes of things to each other. The import of the proposition must be shifted from that of stating the relation of subject and predicate, or of object and attribute, to that of stating the relation of inclusion and exclusion of two classes to each other.

The question therefore at once arises, How do the eight verbal forms just quoted stand in relation to the five which we have seen to be in their own way, exhaustive? As this is a very important enquiry towards a right understanding of the nature and functions of propositions, I shall make no apology for going somewhat into details respecting it. As regards the first five out of the eight, the correspondence is of course complete, if we understand that the word 'some' is to

be understood, as here, to distinctly exclude 'all[1].' But then, if so, what account is to be given of the remaining three out of the eight? Only one account, I think, can be given. They are superfluous or ambiguous equivalents for one or more of the first five. This may need a moment's explanation. By calling the first five complete and unambiguous we mean, as already remarked, that if one of these propositions were uttered, only one of the forms of diagram could be selected to correspond with it; and conversely, if one of the diagrams were pointed out, it could be referred to only one of the verbal expressions. But if we were given one of the latter three to exhibit in a diagram we could not with certainty do so. Take for instance the proposition 'No *A* is some *B*.' If we proposed thus to exhibit it we should find that diagrams (2) and (4) are equally appropriate for the purpose; whence this proposition is seen to be ambiguous and in a minimum schedule, superfluous. Similarly its formal converse the proposition 'Some *A* is not any *B*,' is equally fitly exhibited in diagrams (3) and (4) and therefore appropriately in neither. Consequently it also must be regarded as needless in our scheme. The case of the remaining proposition, 'Some *A* is not some *B*,' is still worse. If 'some' were interpreted as including 'all,' then it is equally applicable to all our five distinct possible cases. If, on the other hand, 'some' stands for 'not all' then we are

[1] Though his language is not always free from ambiguity this is, on the whole, the sense I understand Hamilton to adopt. Sometimes he is quite explicit; *e.g.* "Affirming *all men are some animals*, we are entitled to infer the denial of the propositions *all men are all animals*, *some men are all animals.*" *Logic*, Vol. II. p. 283. The reader will understand that I am not advocating this sense of the word, but merely pointing out that if we propose to adopt a certain schedule of propositions this sense will best serve our purpose in the way of distinguishing them unambiguously.

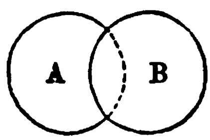
required to ascertain what must be the mutual relations of two classes when assigned by the condition that 'some, but not all, of each shall be distinct from some, but not all, of the other.' This, as Hamilton himself implies, (*Discussions*, p. 691) holds good only of *coincident* classes; so that 'Some *A* is not some *B*' yields the same relation as 'All *A* is all *B*,' though it looks at the relation from another point of view. In this sense the proposition is equivalent to a proverbial form of distinguishing between different members of one common class, as when we say, 'There are gamblers and gamblers,' meaning that some of them are very different from others. This is quite a legitimate interpretation, provided we clearly recognize that we have shifted our ground: what we now indicate is not an objective class relation directly, but a judgment of ours indirectly founded upon that relation rather than intended to display it. It is an intelligible proposition, but belongs to another schedule.

The ambiguity affecting these three last forms is, it need hardly be remarked, reciprocal. That is, so long as these three are retained in the scheme, we should not know, on a diagram being presented to us, which proposition was meant to be exhibited; any more than we can draw the diagram when a proposition is stated. Diagram (3), for instance, might under these circumstances be read off indifferently as 'Some *A* is all *B*,' 'Some *A* is not any *B*,' or 'Some *A* is not some *B*.'

It may perhaps be replied that there is still a use in retaining forms of proposition which thus refer ambiguously to two or more actual class relations, in addition to those forms which refer unambiguously to one only. It may be urged that if we do not know which of the two is really applicable, though one or the other must certainly be so, there is an opening for a form which covers both of them. I do not

think that this will do. In the first place there is the objection that the employment of terms in their extensive signification implies that we are expressing their actual relation to each other in the way of inclusion and exclusion, and not our imperfect knowledge of that relation. At any rate this seems to be so when we make use of diagrams of this kind, for the circles must either cut one another or not do so; we cannot express a diagrammatic ignorance whether they do or not. We may feel a doubt whether they should do so or not, but we must make them do one thing or the other.

An attempt is sometimes made in this way by the device of marking a part of one of the circles with a dotted line only. Thus 'Some *A* is not *B*' would be exhibited as follows:—



(as, for instance, is done, amongst others, by Thomson in his *Laws of Thought*). The dotted part here represents of course our ignorance or uncertainty as to whether the *A* line should lie partly inside *B*, or should entirely include it. But surely, if we are thus ignorant we have no right to prejudge the question by drawing one part inside, even as a row of dots. What we ought to do is to draw *two* lines, one intersecting *B* and the other including it. Doing this, there is no need to dot them; it is simpler to draw at once in the ordinary way the three figures (3) (4) and (5) above, and to say frankly that the common 'Some *A* is not *B*' cannot distinguish between them. In other words this form cannot be adequately represented by one of these diagrams; it implies a different propositional theory. Ueberweg (*Logic, Transl.*

p. 217), has thus represented propositions of this kind by alternative diagrams, as well as by a scheme of dotted lines.

But there is a more conclusive objection than this last. If we were disposed to admit the three latter Hamiltonian forms on such a plea as the one in question, we should be bound in consistency to let in several more upon exactly the same grounds. Take, for instance, the first two, 'All *A* is all *B*,' 'All *A* is some *B*.' We often practically want some comprehensive form of expression which shall cover them both, and this was excellently provided by the old *A* proposition 'All *A* is *B*,' which just left it uncertain whether the *A* was all *B*, or some *B* only. Perhaps this is indeed the very commonest of all the forms of assertion in ordinary use. Hence if once we come to expressing uncertainties or ambiguities we should have to insist upon retaining this old *A*, not as a substitute for one of the two first Hamiltonian forms, but in addition to them both. Similarly we should require a form to cover the first and the third. Or again, we might desire a form to cover all the first four; for we might merely know (as indeed is often the case) that *A* and *B* had some part, we did not know how much, in common. What we should want, in fact, would be a simple equivalent for 'Some or all *A* is some or all *B*;' or, otherwise expressed, a form for merely denying the truth of 'No *A* is *B*:' in other words, 'Some *A* is *B*,' where 'some' is taken in its common sense.

The Hamiltonian schedule has, no doubt, a specious look of completeness and symmetry about it. Affirmation and denial, of some and of all, of the subject and predicate, give clearly eight forms. But on subjecting them to criticism, by enquiring what they really say, we see that this completeness is merely verbal, and objectively speaking

illusory[1]. Regard them as expressing the relations of class inclusion and exclusion, (and this I strongly hold to be the right way of regarding them, though quite aware that it is not the way in which they commonly are regarded), and we only need, or can find place for, *five*. Regard them as expressing to some extent our uncertainty about these class relations, and we want more than eight. This exact group of eight seems merely the outcome of an exaggerated love of verbal symmetry[2].

If indeed our choice lay simply between the old group and the Hamiltonian, the old one seems to me far the soundest and most useful. One or more of those four will express almost all that we can want to express for purely logical purposes, and as they have their root in the common needs and expressions of mankind, they have a knack of signifying just what we want to signify and nothing more. For instance, as above remarked, we may want to say that 'All *A* are *B*,' when we do not know whether or not the two terms are coextensive in

[1] I am speaking, of course, of ordinary logical predication. For this purpose the eight forms are, as remarked, the outcome of a mere love of verbal symmetry. An instructive contrast is yielded by substituting the eight corresponding forms obtained by ringing the changes on such a proposition as this:—All (or some) trains do (or do not) stop at all (or some) stations. Here we have eight appropriate and really distinct meanings; but then we are doing something more here than merely quantifying our predicate.

[2] De Morgan appears to have entertained a similar view, for, after describing the five propositional forms above, he adds "these enunciations constitute the system at which Hamilton was aiming" (*Camb. Phil. Tr.* x. 439).—This and two other papers in the same volume contain the fullest and acutest criticism of the Quantification doctrine which I have anywhere seen. The best compendious account of his own view and criticism of Hamilton's view, is, I think, in the article "Logic" in the *English Cyclopædia*.—Hamilton (*Discussions*, p. 683) has made an attempt to represent his eight forms diagrammatically, on Lambert's linear plan. He offers only six diagrams, and as much as admits that one of these is superfluous. See also his *Logic* (II. 277) where the eight propositions are illustrated by four circular diagrams.

their application. The old form just hit this off. An obvious imperfection in the Hamiltonian scheme is that with all his eight forms he cannot express this very common and very simple form of statement, by means of a single proposition. He can express the less common state of doubt between the two, 'All *A* is some *B*,' and 'Some *A* is some *B*,' by one of what I have termed his superfluous forms, viz., by his 'Some *B* is not any *A*,' for it exactly covers them both.

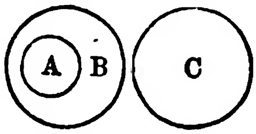
So long as we confine ourselves to the five propositions which correspond to the five distinct diagrams, we are on clear ground. These both rest on a tenable theory as to the import of propositions sufficiently to give them cohesion and make a scheme of them. That theory is, as above explained, that they are meant to express all the really distinct relations of actual class inclusion and exclusion of two logical terms, and none but these.

The advantages of this form of propositional statement, if few, are at any rate palpable and unmistakable. Each form has a corresponding diagram which illustrates its exact signification with the demonstrative power of an actual experiment. If any sluggish imagination did not at once realise that from 'All *A* is some *B*,' 'No *B* is any *C*,' we could infer that 'No *A* is any *C*,' he has only to trace the circles, and he sees it as clearly as any one sees the results of a physical experiment. And most imaginations, if the truth were told, are sluggish enough to avail themselves now and then of such a help with advantage.

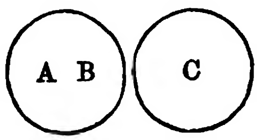
But whilst this is said it ought clearly to be stated under what restrictions such an appeal may fairly be made. The common practice, adopted in so many manuals, of appealing to these diagrams,—Eulerian[1] diagrams as they are often

[1] A brief historic notice is given, in the concluding chapter, about the employment of diagrams in Logic.

called,—seems to me very questionable. Indeed when it is done, as it generally is done, without a word of caution as to the important distinction between the implied theories about the import of propositions, it seems to me that there can be no question as to its being wrong. The old four propositions *A*, *E*, *I*, *O*, do not exactly correspond to the five diagrams, and consequently none of the moods in the syllogism can in strict propriety be represented by these diagrams[1]. We may sometimes see Celarent represented thus:—



But this representation is too narrow. The affirmative here is not 'All *A* is *B*,' but 'All *A* is *some B*.' To represent Celarent adequately in this way we should have to append also the diagram,



representing 'All *A* is all *B*,' 'No *B* is *C*;' and to say frankly that we know that one of these diagrams will represent our syllogism, but we do not know which[2].

[1] This ambiguity of reference, and consequent necessity for a plurality of diagrams, has been fully illustrated by Dr Keynes (*Formal Logic*). One of the few writers,—of those whose works I have seen,—who had previously realized this essential characteristic of the Eulerian diagrams, is Pokorny (*Logik*: 1878).

It may be observed that the *rejected* mood:—'No *B* is *C*; No *A* is *B*,' with the consequent inability to draw a conclusion, can be thus exhibited; because the universal negative is the only form common to the two schemes.

[2] If it be urged that the upper diagram is the general one, including the lower as a special case, the answer is two-fold. First, that this would be tantamount to a rejection of this scheme of propositions; and second, that even so we should not meet the case of syllogisms involving

Of course this inability to represent each syllogistic figure by one appropriate diagram will not always affect their cogency as illustrations. Any one can see in the above instance that one diagram will practically take the place of the other; and so it would in *Barbara,* but not in every instance, as we shall presently shew more in detail. But none the less must we remember that the systems of propositions are really based on distinct theories, and we have consequently no right thus without warning to use the diagrams of one system to represent the propositions and syllogisms of another.

What then is the number and nature of the Syllogistic Figures, if we may still call them such, which we should have to adopt if we adhered with strict consistency to the propositional theory now under discussion? The matter is not one of much importance in itself, but it deserves consideration as serving to impress upon the reader what are the true representative characteristics of the diagrams with whose appearance he is already so familiar; and I will therefore go into it with more detail than would otherwise be needed[1].

particular propositions, as the reader will see if he tries thus to exhibit, say, *Disamis.* (See further on for more illustration of this point.)

[1] When the substance of this chapter was first written out for *Mind* I was unable to ascertain that any attempt had been made to reconstruct the syllogistic figures upon this propositional scheme. I have since found that almost exactly the same results as are given here had been already obtained by F. A. Lange, in his admirable *Logische Studien,* though from a somewhat different point of view. Hamilton (*Logic,* II. p. 475) has given a table of valid Syllogistic Moods adapted to his own complete scheme of propositions. It is also printed in Dr Thomson's *Laws of Thought,* p. 188. Mr Ingleby also (*Outlines of Logic,* 1856), as a disciple of Hamilton, has discussed the problem with more fulness, but his results are naturally widely different from mine, since he too admits all the eight forms as standing on an equal footing as regards admissibility.

The problem of the Syllogism under the conventional rules is briefly this. Given three terms X, Y, Z, our range of liberty extends to making a couple of universal or particular, affirmative or negative, propositions out of them; and then arranging these propositions so that the term which occurs twice over, (the middle term), shall be subject of both, predicate of both, or subject of one and predicate of the other: but we are not allowed to introduce negative[1] terms such as not-X. This yields sixty-four possibilities, from nineteen only of which can a necessary conclusion be drawn.

Take then the properly corresponding conditions for the other scheme now under discussion, and see what we are led to. The very similarity of the conditions leads to differences in application. To begin with, we recognize five forms of proposition instead of four, but then on the other hand we recognize no distinction between subject and predicate. Hence we have only twenty-five possible combinations of premises, instead of sixty-four, to submit to examination. Again, there is another very important ground of distinction arising out of the fact (already adverted to) that on this scheme 'some' does not include 'all,' but is incompatible with it. That is to say the two conclusions 'Some X is Z' 'All X is Z' are not here compatible with each other. These points will come out better in the course of the following discussion.

Make then an arrangement of all the possible combinations of premises with a view to determining how many of them will lead to certain and unambiguous conclusions. They will stand as follows:—

[1] The rejection of this convention, with the consequent free introduction of negative subjects and predicates, is one of the most marked characteristics of De Morgan's System.

Premise I.	*Premise* II.
All *Y* is all *Z*,	All *X* is all *Y*,
All *Y* is some *Z*,	All *X* is some *Y*,
Some *Y* is all *Z*,	Some *X* is all *Y*,
Some *Y* is some *Z*,	Some *X* is some *Y*,
No *Y* is any *Z*.	No *X* is any *Y*.

Now combine each of the first column with each of the second column, and take the results in detail :—

1. All *Y* is all *Z*, All *X* is all *Y*. This yields the conclusion, All *X* is all *Z*. [See Figure (1), p. 22.]

2. All *Y* is all *Z*, All *X* is some *Y*. This yields the conclusion, All *X* is some *Z*. [Fig. (2.)]

3. All *Y* is all *Z*, Some *X* is all *Y*. This yields the conclusion, Some *X* is all *Z*. [Fig. (3.)]

4. All *Y* is all *Z*, Some *X* is some *Y*. This yields the conclusion, Some *X* is some *Z*. [Fig. (4.)]

5. All *Y* is all *Z*, No *X* is any *Y*. This yields the conclusion, No *X* is any *Z*. [Fig. (5.)]

6. All *Y* is some *Z*, All *X* is all *Y*. This yields a valid conclusion, but may be rejected on the ground that it is formally identical with No. (3), merely substituting *Z* for *X*, and *vice versâ*.

7. All *Y* is some *Z*, All *X* is some *Y*. This yields the conclusion, All *X* is some *Z*. [Fig. (6).]

8. All *Y* is some *Z*, Some *X* is all *Y*. It may seem as if the conclusion that common logic would draw 'Some *X* is *Z*,' ought to hold good here. But the peculiar signification of 'some' forbids a conclusion; for any one of the four (with us, not merely distinct but mutually hostile) propositions 'All *X* is some *Z*,' 'Some *X* is all *Z*,' 'All *X* is all *Z*,' 'Some *X* is some *Z*,' are equally compatible with the premises. Hence no single conclusion is admissible.

9. All *Y* is some *Z*, Some *X* is some *Y*. The state of

things here is similar to that in the last case. We reject the syllogism, on the ground of its admitting of two mutually inconsistent conclusions 'All X is some Z,' and 'Some X is some Z.'

10. All Y is some Z, No X is any Y. Rejected on the same ground as the last two, inasmuch as it admits of three mutually hostile conclusions. Common logic agrees with us in this rejection, because two of the possible conclusions, viz.: 'No X is any Z,' 'Some X is some Z,' are also recognized by it as hostile, whereas in cases (8) and (9) it recognized no such hostility.

11. Some Y is all Z, All X is all Y. Admissible in itself, but dismissed on the ground of its formal identity with (2).

12. Some Y is all Z, All X is some Y. Rejected (as in common logic) because *any* of the five possible conclusions is compatible with the premises.

13. Some Y is all Z, Some X is all Y. Formally identical with No. (7).

14. Some Y is all Z, Some X is some Y. Rejected as leading to three possible conclusions. Common logic concurs here.

15. Some Y is all Z, No X is any Y. This yields the conclusion, 'No X is any Z.' [Fig. (7.)]

16. Some Y is some Z, All X is all Y. Formally identical with (4).

17. Some Y is some Z, All X is some Y. Rejected as leading to three conclusions: formally identical with (14).

18. Some Y is some Z, Some X is all Y. Common logic would treat this as *Disamis*, with conclusion 'Some X is Z;' but we have to reject it, because we do not know whether this some X is 'all Z,' or 'some Z,' which with us are conflicting conclusions. Formally identical with (9).

19. Some *Y* is some *Z*, Some *X* is some *Y*. Rejected as admitting of five possible conclusions.

20. Some *Y* is some *Z*, No *X* is any *Y*. Rejected as admitting three conclusions.

21. No *Y* is any *Z*, All *X* is all *Y*. Formally identical with (5).

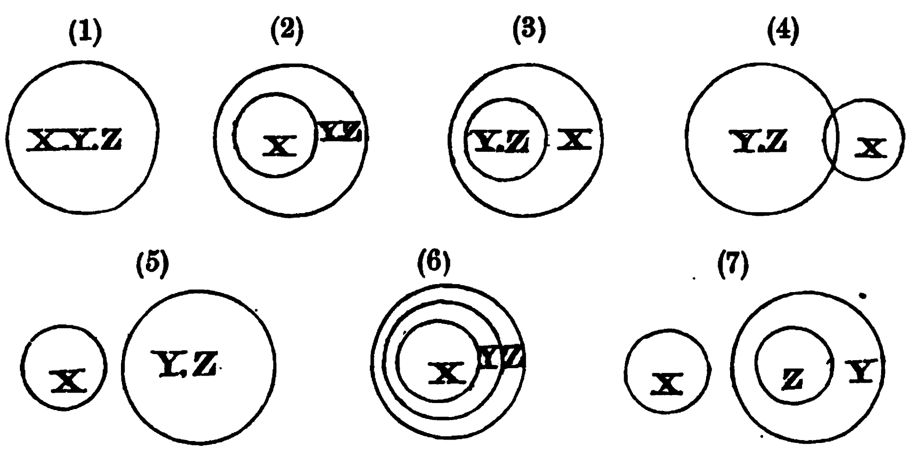
22. No *Y* is any *Z*, All *X* is some *Y*. Formally identical with (15).

23. No *Y* is any *Z*, Some *X* is all *Y*. Common logic would conclude 'Some *X* is not *Z*,' but since this proposition covers three of our distinct forms, we have to reject the syllogism as leading to no certain conclusion. Formally identical with (10).

24. No *Y* is any *Z*, Some *X* is some *Y*. This case corresponds to the last; we reject it on the same ground. It is formally identical with (20).

25. No *Y* is any *Z*, No *X* is any *Y*. Rejected, as admitting of five possible conclusions. Common logic of course concurs in this rejection, since it interprets universal negatives exactly as we do.

It will be seen therefore that we should admit seven forms or 'moods' of syllogism as distinct, and only seven. They are thus exhibited in diagrams:



A study of these figures will, I think, convince any one, independently of detailed investigation, that they represent all the ways in which each of two figures, say circles, can stand in relation (relation, that is, of the five specified kinds) to a third such figure, so that their mutual relation to each other shall be unambiguously determined thereby. And this mediate and unambiguous determination is exactly what I understand that the syllogism proposes to effect here.

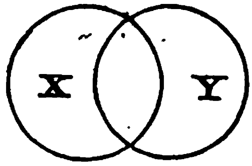
So much then for this second scheme of propositional import and arrangement. In spite of its merit of transparent clearness of illustration of a certain number of forms of statement, it is far from answering our purpose as the basis of an extension of Logic. Its employment soon becomes cumbrous and unsymmetrical, and possesses no flexibility or generality. Fortunately there is another mode of viewing the proposition, far more powerful in its applications than either of those hitherto mentioned. It is the basis of the system introduced by Boole, and could never have been invented by any one who had not a thorough grasp of those mathematical conceptions which Hamilton unfortunately both lacked and despised. The fact seems to be that the moment we quit the traditional arrangement and enumeration of propositions we must call for a far more thorough revision than that exhibited on the system just discussed. Any system which merely exhibits the mutual relations of two classes to each other is not general enough. We must provide a place and a notation for the various combinations which arise from considering three, four, or more classes; in fact we must be prepared for a thorough generalization. When we do this we shall soon see that the whole way of looking at the question which rests upon the mutual relation of classes, as regards exclusion and inclusion, is insufficient. There is a fatal cumbrousness and want of

symmetry about it which renders it quite inappropriate for any but the simplest cases.

A way of interpreting and arranging propositions which may be substituted for both the preceding (for the purpose of an extended Symbolic Logic), is perhaps best described as implying the occupation or non-occupation of compartments. What we here have to do is to conceive, and invent a notation for, all the possible combinations which any number of class terms can yield; and then find some mode of symbolic expression which shall indicate which of these various compartments are empty or occupied, by the implications involved in the given propositions. This is not so difficult as it might sound, since the resources of mathematical notation are quite competent to provide a simple and effective symbolic language for the purpose. What we can afford to say about this scheme here is of course merely preliminary, since many questions will arise which will demand discussion and illustration in the following chapters. Enough, however, may easily be said to bring out clearly its bearing on the particular subject which has just been discussed, viz. the number of distinct forms of proposition which ought to be recognized. The view which is here taken is still distinctly a *class* view rather than a predication view; but, instead of regarding the mutual relation of two or more classes in the way of inclusion and exclusion, it substitutes a complete arrangement of all the subdivisions which can be yielded by putting any number of classes together, and indicates whether any one or more of these classes is occupied; that is, whether things exist which possess the particular combination of attributes in question.

A fair idea of the meaning, scope and power of this system will be gained if we begin with two class terms X and Y, and consider the simple cases yielded by their combination.

It is clear that we are thus furnished with four possible results, or compartments, as we shall often find it convenient to designate them; for everything which exists must certainly



possess both the attributes marked by X and Y, or neither of them, or one and not the other. This is the range of possibilities, from which that of actualities may fall short; and the difference between these two ranges is just what it is the function of the proposition to indicate. We will confine ourselves at present to intimations that such and such compartments are *empty*, since this happens to be the simplest alternative. Now how many distinct cases does this system naturally afford?—we must approach it, let us remember, without prepossessions derived from any customary divisions and arrangements of propositions.

We should naturally be led, I think, to distinguish fifteen different cases on such a system as this, and these would fall into four groups. For there may be one compartment unoccupied, which yields four cases; or two unoccupied, which yields six cases; or three unoccupied, which yields again four cases; or none unoccupied, which yields but one case. *All* cannot be unoccupied, of course, for we cannot deny both the existence and the non-existence of a thing; or, to express it more appropriately on this scheme, given that a thing exists it must be put somewhere or other in our all-comprehensive scheme of possibilities.

As this whole scheme will be thoroughly worked out in future chapters, I will only call attention here to the nature of the four simplest out of these fifteen cases. Writing, for

simplicity, $\bar{x}$ for not-x and $\bar{y}$ for not-y, these four would be thus represented:—

$$\begin{aligned} xy &= 0, \text{ or No } x \text{ is } y, \\ \bar{x}y &= 0, \text{ or All } y \text{ is } x, \\ x\bar{y} &= 0, \text{ or All } x \text{ is } y, \\ \bar{x}\bar{y} &= 0, \text{ or Everything is either } x \text{ or } y. \end{aligned}$$

On this plan of notation xy stands for the compartment, or class, of things which are both x and y; and the equation $xy = 0$ expresses the fact that that compartment is unoccupied, that there is no such class of things. And similarly with the other sets of symbols.

A moment's glance will convince the reader how entirely distinct the group of elementary propositions thus obtained is from that yielded by either of the other two schemes; though, viewed from its own appropriate standpoint, it is just as simple and natural as either of them. One of the four forms is of course the universal negative, which presents itself as fundamental on all the three schemes. Two others are universal affirmatives, but with the subject and predicate interchanged. But the fourth is significant, as reminding us how completely relative is the comparative simplicity of a propositional form. On the present scheme this is just as simple as any of the others; but in the traditional arrangement it would probably obtain admittance only as a disjunctive, since that arrangement hates the double negation 'No not-x is not-y' and does not like even the simpler 'All not-x is y.' Indeed on hardly any other view than that now before us could such a verbal statement as this last be considered an elementary one; many persons would have to put it into other words before clearly understanding its import.

We might easily go through the eleven remaining cases referred to above, but to do so would be an anticipation o

the discussions of future chapters. It may just be noticed, however, in passing, that the emptying out (as we may term it) of two compartments does not necessarily give a proposition demanding a greater amount of verbal statement than that of one only does. For instance the combination $x\bar{y}=0$, $\bar{x}y=0$, expresses the coincidence of the two classes x and y; it is 'All x is all y.' That of $xy=0$, and $\bar{x}\bar{y}=0$, yields the statement that x and y are the contradictory opposites of one another, that x and not-y are the same thing, and consequently y and not-x. These and other results of this system will call for attention hereafter.

The full merits of this way of regarding and expressing the logical proposition are not very obvious when only two terms are introduced, but it will readily be seen that some such method is indispensable if many terms are to be taken into account. Let us introduce three terms, x, y, and z; and suppose we want to express the fact that there is nothing in existence which combines the properties of all these three terms, that is, that there is no such thing as xyz. If we had to put this into the old forms we should find ourselves confronted with six alternative statements, all of them tainted with the flaw of unsymmetry; viz. No x is yz, No y is xz, No z is xy, and also the three converse forms of these. No reason could be shown for selecting one rather than another of them; and if we attempted to work with the symmetrical form 'There is no xyz,' we should find that we had no supply of technical rules at hand to connect it with propositions which had only x, y, or z, for subject or predicate.

If we tried the second propositional theory we should only reduce the above six unsymmetrical alternatives to three; three being got rid of by our refusing to recognize that conversion makes any difference in the proposition. But the same inherent vice of a choice of unsymmetrical alterna-

tives would still confront us. No reason could be given why we should express the facts by saying that the class x excludes the class yz, rather than that the class y excludes that of xz, or z that of xy. Common language may be perfectly right in accepting such a state of things; but a sound symbolic method ought to be naturally cast in a symmetrical form if it is not to break down under the strain imposed by having to work with three or more terms. This requires us to avoid such forms as 'the class x excludes that of yz,' as well as all the statements of the common Logic, and to put what we have to express into the symbolic shape $xyz = 0$. The verbal equivalents for this are, of course, that there is no such thing as xyz, or that the compartment which we denote by xyz is empty.

It deserves notice that ordinary language does occasionally recognize the advisability of using symmetrical expressions of this kind, though the common Logic shows no fondness for them. We should as naturally say, for example, that 'cheapness, beauty and durability never go together,' or that 'nothing is at once cheap, beautiful, and durable,' as we should use one of the forms which divide these three terms between the subject and the predicate. But this latter plan is what would be adopted presumably by the strict logician, ~~by~~ his arranging it in some such form as 'no cheap things are beautiful-durable.' It need not be remarked that popular language, though occasionally making use of such symmetrical forms, has never hit upon any general scheme for their expression, and would be sadly at a loss to work upon more complicated materials. Especially would this be the case where negative predicates or attributes had to be taken into account as well as positive. However what we are here concerned with is the insufficiency of the ordinary logical view rather than the occasional ingenuity of popular expression.

In these remarks I have endeavoured to keep rather closely to the enquiry suggested at the outset; that, namely, of the principal varieties of fundamentally distinct logical forms, and the foundation on which each different arrangement must be understood to rest. It seems quite clear that no attempt can be made to answer this question until we have decided, in a preliminary way, what view we propose to take of the proposition and its import. There is no occasion whatever to tie ourselves down to one view only, as if the import of propositions were fixed and invariable. Other views might have been introduced in addition to the three which have been thus examined; though these appear to me to be the only ones with which the student is likely to have to make much acquaintance.

Each of the three stands upon its own basis, yields its appropriate number of fundamentally distinct propositions, and possesses its own merits and defects. The old view has plenty to say for itself, and for ordinary educational purposes will probably never be superseded. It is very simple, it is in close accordance with popular language, and it possesses a fine heritage of accurate technical terms and rules of application. Its defects seem to me to be principally these:—that it does not yield itself to any accurately correspondent diagrammatic system of illustration; and that its want of symmetry forbids its successful extension and generalization.

The great merit of the second plan, or that of class inclusion and exclusion, is its transparent clearness of illustration. We may be said thus to *intuite*[1] the proposition. This has indeed caused what must be regarded as an unwarranted

[1] F. A. Lange (*Logische Studien*, p. 9) maintains that such intuitions of space lie at the bottom of our logical axioms exactly as they do in the case of our geometrical axioms.

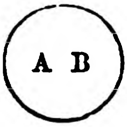
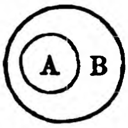
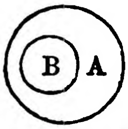
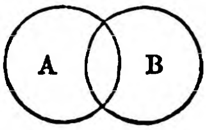
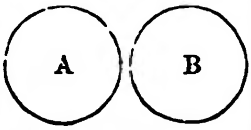
employment of its diagrams by those who do not realize that its **five distinct** forms of proposition cannot be properly fitted in with the four of the traditional scheme. This clearness is however almost its only merit. It possesses little more of symmetry, or consequent adaptability to generalization, than the former, and it is considerably further removed from popular forms of expression. Above all, as it insists upon exhibiting the *actual* relations of the two classes to each other, it has no power to express that degree of ignorance about these relations which **many** propositions are bound to display, if they are to state **all that** we know, and nothing but what we know, about the **relation** of the subject and predicate to each other. Hamilton's eight propositions seem to me an inconsistent and partial attempt to remedy this latter defect, in so far as the last three of his forms are concerned.

The third scheme is, of course, in comparison with the others, an artificial one, and possesses the merits and defects which might be expected in consequence. It is couched in too technical forms, and is too far removed from the language of common life, for it ever to become a serious rival of the traditional scheme on the ground appropriate to this latter. In respect of symmetry, however, and the power which comes of symmetry, it is not easy to see what can be put into competition with it, as will abundantly appear in the course of the following chapters.

The following comparative table will aid the reader in keeping in mind the distinctions insisted on in the course of this chapter.

In the first column the five possible distinct objective class relations are exhibited. In the second column stand the groups of ordinary propositions requisite for the purpose of unambiguously expressing each of these relations. In the

third, we express these relations by single propositions with quantified predicates, and a special interpretation of '*some.*'

(i) Diagrammatic	(ii) Common Logic	(iii) Quantified	(iv) Symbolic
	All *A* is *B* All *B* is *A*	All *A* is all *B*	$\left.\begin{array}{l}A\bar{B}=0\\ \bar{A}B=0\end{array}\right\}$
	All *A* is *B* Some *B* is not *A*	All *A* is some *B*	$\left.\begin{array}{l}A\bar{B}=0\\ \bar{A}B=v\end{array}\right\}$
	All *B* is *A* Some *A* is not *B*	Some *A* is all *B*	$\left.\begin{array}{l}\bar{A}B=0\\ A\bar{B}=v\end{array}\right\}$
	Some *A* is *B* Some *A* is not *B* Some *B* is not *A*	Some *A* is some *B*	$\left.\begin{array}{l}AB=v\\ A\bar{B}=v\\ \bar{A}B=v\end{array}\right\}$
	No *A* is *B*	No *A* is any *B*	$AB=0$

In the fourth we represent them symbolically. (As regards the signification of the latter expression (>) I am anticipating here. As will be shown in chap. VII., $A\bar{B}>0$ means 'There is *A* which is not *B*,' or '$A\bar{B}$ is something,' *i.e.* is not to be $=0$, or obliterated.)

In the above arrangement the last three columns are forced into following the lead of the first. To see into what shape they will throw themselves when this pressure is removed, we may adopt another arrangement; in which they all, so to say, think of nothing but following out their own bent and framing themselves as simply and symmetrically as

possible. In this case the second column contracts into the familiar four propositions; the third, when regarded as consisting of verbal statements unchecked by diagrams, shows a decided inclination towards mere grammatical symmetry, and will tend to spread out into Hamilton's redundant eight-fold scheme; whilst the last yields its own compendious table to be explained in a future chapter.

II.	III.	IV.
	All A is all B.	$AB = 0, > 0$.
All A is B.	All A is some B.	$A\bar{B} = 0, > 0$.
Some A is B.	Some A is all B.	$\bar{A}B = 0, > 0$.
No A is B.	Some A is some B.	$\bar{A}\bar{B} = 0, > 0$.
Some A is not B.	No A is any B.	
	No A is some B.	
	Some A is not any B.	
	Some A is not some B.	

CHAPTER II.

SYMBOLS OF CLASSES AND OF OPERATIONS.

IN Symbolic Logic we are concerned with two kinds of symbols, which are commonly described as standing respectively for *classes* and for *operations*. It should be remarked, however, that these two kinds cannot be very sharply distinguished from one another, inasmuch as each to some extent implies the other. Thus a class may be almost always described as the result of an operation, namely of an operation of selection. The individuals which compose the class have been somehow taken from amongst others, or they would not be conceived as being grouped together into what we call a class. Similarly what we call operations always result in classes; at least all the operations with which we are here concerned will be found to do so. The mere signs of operations never occur by themselves, but only in their applications, so that practically we never encounter them except as yielding a class, and in fact almost indistinguishably merged into a class[1]. For purposes of exposition, however, it

[1] Popular usage does not seem perfectly clear upon this point even in mathematics, since it is frequently overlooked that what we commonly see expressed are not operations but *results* of operations. Thus $\sqrt{2}$ stands for a number pure and simple, that is, for the number which when squared

will be necessary to make the distinction, since those logical operations by which classes are governed, and which we shall presently discuss, being less familiar to the reader than the direct symbols for simple classes, will demand far the greater share of our attention.

The former kind of symbols, namely those which stand for classes of things, need occupy us but a very brief space here, since we may presume that the reader has already become familiar with them elsewhere. Nothing is more usual than to put such general symbols as X and Y in place of the concrete subjects and predicates of our propositions. In fact it is necessary to do this when we wish to judge of any kind of general rule upon its merits, and to keep clear of individual circumstances and of misleading associations. Hence we adopt the practice of putting single letters to stand for whole classes of individuals, in other sciences, as in stating Law cases, and indeed in some of the circumstances of ordinary life.

What indeed the logical student may have to do in this direction is not so much to acquire new associations as to divest himself temporarily of some old ones. For instance we have, strictly speaking, no concern whatever within the limits of our enquiry with such distinctions as those between denotation and connotation, between essential and accidental attributes, or even with that between subject and predicate. Not of course that we would for a moment imply that such

will yield 2. We imply this when we say $\sqrt{2} = 1.41...$, for of course we could not equate an operation to a numerical result. That is, $\sqrt{2}$ denotes a mere number but indicates at the same time the operation by which that number is justified. Exactly so in Logic. We must understand xy, $\frac{x}{y}$, &c. strictly to denote mere classes, but to indicate by their form the nature of the operation by which the classes in question are to be justified or obtained, as described in the course of this and the following chapter.

distinctions as these are unimportant. Quite the reverse; the study of them furnishes decidedly one of the most valuable educational advantages to be derived from the Common Logic. We omit them solely in order to keep our own Science homogeneous and symmetrical. I wish to insist upon this point strongly because it has been so frequently overlooked. Symbolic Logic is not a generalization of the Common Logic in all directions alike. It confines itself to one side of it, viz. the class or denotation side,—probably the only side which admits of much generalization[1],—and this it pushes to the utmost limits, withdrawing attention from everything which does not develope in this direction. Moreover, inasmuch as it is a purely Formal Science, the resolve to fit it in with the problems of Induction, or to regard it as an introduction to the Principles of Science in general, seems to me a grave error, and to result merely in the attempt to combine heterogeneous materials.

We shall find it convenient therefore to say that we regard the literal symbols, x, y, z, &c., as standing simply for classes of individuals, no matter how these classes are determined. For instance, it is of no consequence, for our purposes, whether the things which we denote by x and y are actually marked out to us by a substantive or by an adjective; by reference to their essential or to their accidental attributes; by a general connotative term, by a merely denotative term, or by some purely arbitrary selection of a number of individuals. Of course, such classes must be somehow distinguished or distinguishable from others, or the symbols would not be significant. If I am to make use of the terms x and y to any purpose, I must obviously have some means of making it clear to myself and to others which things are x and which

[1] Some justification of this statement will be offered in a future chapter.

are not, which are y and which are not. But all discussion as to *how* this is done, and all analysis of the grounds and processes of doing it, deeply interesting as such discussions are when philosophically regarded, must just now be relegated to Common Logic, to Grammar, or to Psychology. With us, x and y regarded as class terms must be considered as having mere denotation, as standing for certain assignable individuals or groups of individuals[1].

Suppose, for instance, I make the assertion, "The Philosophical writers whom I have in view are the à posteriori school, the first editor of the *Edinburgh Review*, and the two Mills." Here I have occasion to allude to four classes, as we may term them. The first is indicated by a many-worded term 'the philosophical writers whom I have in view.' The class here, it will be observed, is partly determined by the connotative terms 'philosophical writers,' but these are identified in some way known perhaps only to myself; for possibly no one else may at the moment know *how* I recognize or keep in mind this group, or what are the common characteristics which I detect throughout it. When therefore I put a symbol, say w, to represent this many-worded term, such symbol does not exactly correspond to any distinct kind of class term recognized in Logic. Again, as regards "the à posteriori school"; here we have a connotative general name of the familiar kind; this we equally represent by a symbol, say x.

[1] The above account will do as a preliminary one, but the reader will find in the sequel that it needs modification or enlargement in two respects. First (as explained in ch. VI.) it is imperatively necessary to regard these classes as *hypothetical*, i.e. to lay it down that the employment of a class term does not imply that there are actual existent members of it. Secondly the ordinary conception of a class, as consisting of a plurality of individuals, will need enlargement; it must cover the case, e.g. of the truth or falsity of a single proposition, amongst other things (this is explained in ch. XVIII.).

"The first editor of the Edinburgh" is partly a mere proper name, partly connotative, partly limited arithmetically. And finally "the two Mills" is purely denotative, the class being marked out by a proper name only. But on our system all these important distinctions have to be left out of sight. The classes, whether plural or individual, are all alike represented denotatively by literal symbols, w, x, y, z; and accordingly we equate (as will presently be explained) the first group and the last three. We express the given statement in some such form[1] as $w = x + y + z$.

So much then at present as regards symbols of classes. We must now turn to consider the symbols of operations[2]. In other words, given that we have our various classes thus

[1] Jevons undertook in his system of Logic to make the symbols stand for the *meaning* of the terms and not for their extension. Amongst other evils of such a plan it leads up to the catastrophe of having to maintain "that the old distinction of connotative and non-connotative names is wholly erroneous and unfounded," and that "singular, proper, or so-called non-connotative terms, are more full of connotation or meaning in intent or quality than others, instead of being devoid of such meaning." (*Pure Logic*, p. 6.) It is really impossible to carry out such a view as this; 'particular propositions,' (with which however we have not so much to do) simply refusing in many cases to be interpreted otherwise than in reference to extension, and violence being demanded thus to interpret universal propositions when 'accidental.' F. A. Lange (*Logische Studien*, p. 56), has spoken of the "heilloseste Verwirrung" which has been introduced in this part of the subject since the old scholastic logic of comprehension began to give way, but inconsistently, to a logic of extension. This latter he considers to be essentially the modern view; it is certainly the view we must take in the generalized Symbolic Logic.

[2] The introduction of these symbols marks the real turning point in Symbolic Logic. The distinction was long ago noted by Leibnitz, and by Lambert. The latter (*Neues Organon*, II. 26, 27) distinguishes between "Zeichen der Begriffe", and the "Verbindungskunst der Zeichen." He admits not having (in that work) carried out the latter, but having used letters to express concepts, and words to express their conditions and relations.

designated before us, in what different relations can they be supposed to stand towards each other? How can they be combined or otherwise worked up into new material?

There would appear to be three distinct modes in which we might approach the consideration of this part of the subject. (1) We might resolve to take, as our starting point, the ordinary forms of language; either in their rude shape as exemplified in common speech, or as they appear after they have been trimmed by the logician; and enquire what divisions and arrangements these naturally suggest. Or, (2) we might start by borrowing a set of symbols which have been already adopted in another science (say mathematics), and, taking these provisionally as the basis of our arrangement, examine whether we could conveniently, by analogy or generalization, extract any sort of logical interpretation out of them. Or, (3) we might resolve to start anew for ourselves, by ascertaining what are the really distinct processes, in the way of mutual relations of classes, which we actually have occasion to employ in thinking and reasoning.

There appear to be strong objections to the two former plans. The first, which might seem in some respects the most natural, is rendered impracticable by the extreme laxity and consequent confusion of popular language on almost every point where logical distinctions are involved. Such characteristics of language are intelligible enough when we remember its historic development, and the varied purposes which it has to fulfil besides that of mere logical predication. But they naturally interpose serious difficulties across the path of the logician. If we attempt to carry out a scientific arrangement of valid and distinct meanings by examining the various phrases through the medium of which our meanings are popularly expressed, we shall find that we have

chosen a very troublesome path. Nor have we had as much aid from the ordinary logician here as we might have fairly expected. Several technical points of importance,—of which one, viz. the mutual exclusiveness or otherwise of alternatives, will almost immediately come under our notice,—could be mentioned, upon which the logician has either no settled convictions of his own, or, if he does entertain any, he has scarcely made a serious attempt to guide popular usage into accordance with them.

The objection to the second course, indicated above, is that it is not strictly a logical one. We do not want here to concern ourselves with mathematical relations and symbols, at least not primarily, but with logical relations and their appropriate symbolic representations. Doubtless we shall soon find that the symbolic statement of the latter kind of relation may be conveniently carried out by the use of symbols borrowed from mathematics. But this is a very different thing from starting with these, and trusting to being able to put some logical interpretation upon, say, *plus* and *minus*, or upon the signs indicative of multiplication and division.

The course therefore which it is here proposed to adopt, is the following. We shall begin by examining successively the principal really distinct ways in which classes or class terms practically have to be combined with each other for logical purposes. We shall then proceed to discuss in each case the various words and phrases which are popularly employed to express these combinations, enquiring whether they may not be briefly and accurately conveyed by help of such symbols as those of mathematics. It must however be once for all insisted on, that our procedure is to be logical and not mathematical. For suggestions indeed, coming whether from mathematics or from any other source, we shall be

grateful, (the sign for division is suggested in this way, as will be shown in the next chapter). But our determination and justification of the requisite processes must be governed solely by the requirements of logic and common sense.

As regards the mutual exclusiveness of alternatives, there are three distinct questions involved: (1) that of popular usage; (2) that of logical rule or convention; (3) that of symbolic propriety or convenience[1].

The question of popular usage is, I apprehend, simply this. When two classes are conjoined, by our saying that '*A* is *X* or *Y*,' or that '*B* comprises the *X*'s and the *Y*'s,' can we tell, context apart, whether there is any common part (*XY*) which is intended to be included? It is of no avail to offer examples in which *X* and *Y* are known to be exclusive (e.g. colours), or examples drawn from a subject in which exclusive terms are generally employed. For instance, if any one who is not a botanist were told that all flowering plants, of a certain order, are hypogynous or epigynous, he would have a strong suspicion that, inasmuch as science mostly deals with species, and other exclusives, the case in question would follow the ordinary precedent. But this is not a fair instance. The real question is whether, when we know nothing about the range of the terms employed, or know that they certainly do overlap in their general application, the mere use of the disjunctive form raises the slightest presumption as to how they are related in this respect in the

[1] These three questions seem to me to be confused together by Jevons, who actually charges Boole with transgressing a "self-evident law of thought" (*Pure Logic*, p. 83) on the ground of his adopting,—as a method of logical procedure,—the exclusive plan of notation. It is absolutely necessary to keep apart the question of actual fact (whether of current convention or of technical rule) and the question of convenience of procedure.

particular case in question. If, for instance, we are told that in some unhealthy district 'all the inhabitants are afflicted with consumption or ophthalmia,' we do not, so far, gain a hint as to whether any of the people suffer from both diseases. If it were desired to emphasize the fact that some persons had both, this would be added as a further clause; but it would only be because this fact was left in doubt, not because the contrary was implied. We should adopt the same precaution if we wished to emphasize the opposite fact, viz. that none of the people suffered from both diseases. I feel quite confident that all which popular thought insists upon is that the two, or more, alternatives should cover the whole ground: not that each should also remain clear of the ground of the other.

The conventions of the Common Logic on this subject present no difficulty, though they do present some disagreement and inconsistency. The great majority of logicians have held the popular view just mentioned; and have therefore decided that from the denial of either of two alternatives the affirmation of the other may be inferred, but that from the affirmation of either no denial can be inferred. Not a few however, of whom Hamilton is the best known in England, have laid it down that all disjunctives are to be regarded as exclusive: that is, when we say that 'All *A* is *X* or *Y*,' we are not only justified in inferring that any *A* which is not *X is Y*, but also that any *A* which is *X is not Y*. Strange to say these writers have not generally taken the next step, which consistency obviously demands, of being prepared so to phrase their alternatives as to make them exclusive. No suitable notation is proposed. So long, of course, as the classes *X* and *Y* are mutually exclusive, either in their general or special application, no harm follows: but when they overlap, the validity of the above inferences is

affected. In order to infer the denial of one alternative from the assertion of the other, we must phrase our proposition in the form, '*A* is *X* or *Y*-not-*X*,' or in the form, '*A* is *X*-not-*Y* or *Y*-not-*X*.' As it does not seem that such a postulate is generally made, we can only suppose that these writers do not propose to introduce two terms into a disjunction unless these happen as a matter of fact to be exclusive; and that they thus abandon a large mass of familiar and perfectly justifiable popular assertion. If, for instance, a committee were composed for examining the claims of those interested in a railway, and we were told that all the members were to be either bondholders or shareholders; it would never occur to any one to conclude that those persons who were both bond and shareholders were to be excluded. A logician is quite at liberty to say that he will only employ mutual exclusives in his disjunctions; but, if so, he is bound to recognize that, the terms he employs often actually overlapping, he must cast his disjunctions into some mutually exclusive form. But this certainly has not been generally done.

There now remains the question of our symbolic procedure. Before deciding upon a notation we must examine what are the sorts of distinctions which we desire to note.

I. In the first place, then, we often require to group two or more classes together, so as to make one aggregate class out of them. We do not want to sink their individualizing characteristics, so as to reduce them to one miscellaneous and indistinguishable group; but leaving their respective class distinctions untouched, to throw them together, for some special purpose, into a single aggregate. We want to talk or think of them as a whole. Besides making assertions, for instance, about clergy, lawyers and doctors separately, I may want to make assertions about all three classes together, under the title, say, of the learned professions.

There ought not, one would think, to be much opening here for doubt and confusion. What however with the ambiguities of popular language, and the disputes of rival modes of symbolic statement, a little cloud of confusion has been stirred up. The possibility of this has arisen mainly from a want of proper care in clearly distinguishing between the various questions at issue.

(1) To begin with, there being members common to two classes, we may have it in view to *exclude* these from our aggregate. In physical problems it may happen that one or other alone of two causes will produce a certain effect, but that the two together will either neutralize each other, or by their excess produce some quite different effect. Or, to take a familiar example of another kind, we might have it in view to announce the pardon of two classes of offenders, but expressly wish to exclude the aggravated cases which fall under both heads.

(2) The more frequent case, however, is that in which the common members are included in our formula, so to say, by a double right. Whenever we are discussing mere qualifying characteristics, without introduction of any quantifying circumstances, it would be taken for granted that those who are members of both classes are of course included.

(3) But there is still a third possible case for examination. May we ever want to reckon this common part *twice over?* In numerical applications of our formula, or whenever considerations of quantity are in any way introduced by them, it seems to me quite clear that we may have to do so. And since the simple forms of language indicative of class aggregation are meant to be of general application, many cases might be conceived in which they do thus doubly reckon the common members. This has been too hastily objected to by urging, for example, that if we take all

the cattle and all the beasts of burden from a promiscuous assemblage of animals, we do not think of counting the common part, viz. those which fall under both designations, twice over. Of course we do not, because the word "take" is one which in such an application negatives the possibility of repeated performance. A thing taken once may be considered to be taken altogether. A still better instance in support of this view, so far as material considerations are concerned, would have been found in the proposal to *kill* the members of both of these classes; for some of the beasts of burden, having been put an end to whilst we were dealing with the cattle, would certainly not need any further attention. On the other hand it would be easy to find instances in which the same form of class aggregation by no means denies, but rather suggests, this double counting. Suppose, for instance, we found, by putting together two Acts of Parliament, that all poachers and trespassers are to be fined 20 shillings: is it quite certain that poachers who trespass, could not be fined 40 shillings? This is, I apprehend, a question for the lawyers to decide[1]. But the language is, to the common understanding, certainly ambiguous, which is sufficient for our present purpose. Or, if postmen and parish clerks were authorized to apply for a Christmas-box of five shillings, does anyone suppose that postmen who happened to be parish clerks would not apply for ten shillings altogether; and is it quite certain that their claims would be rejected?

[1] There seems to be no general rule here. On enquiring of a good authority whether, for instance, if it were enacted that auctioneers and hawkers should take out a licence, those who belonged to both classes would require *two* licences, I was told that it would probably be decided by such considerations as whether the conditions of the two licences were the same, whether they were issued by the same authority, were imposed in the same Act, and so forth.

There being, then, these three possible modes of class aggregation, the question arises how they should be respectively indicated. If all three were really in frequent use, there would be very little doubt, I think, that we should have to mark them as follows,

$$A \text{ not-}B + B \text{ not-}A, \quad A + B \text{ not-}A, \quad A + B.$$

In the first we exclude the AB members; in the second we simply count them; in the third we count them twice.

As a matter of fact, however, the third case belongs to Applied rather than to Pure Logic: it is only wanted in certain numerical applications. When we are dealing with problems which involve the counting of the individuals included by our terms,—as, for instance, in such examples as those discussed by Jevons in his *Studies*, under the head of Numerical Logic, or by Dr Macfarlane, in his *Algebra of Logic*,—then there can be no doubt that $A + B$ counts the AB's twice over. As, however, such examples are very exceptional, we may lay them aside, and consider that we have, in Pure Logic, to deal with the two former methods only of aggregation.

The question then becomes simply one of procedure and notation. When we want to represent the aggregate class 'A and B,' shall we expressly exclude double counting by writing it $A + B$ not-A, or shall we write it $A + B$, with the proviso, made once for all, that this shall not be considered as counting AB twice over? As just remarked, the question is one of procedure only, but of procedure in a very wide sense; for much more is at issue here than the trifling point considered a few pages back as to what sort of inferences are yielded by a simple disjunctive. But at the same time, it must be remembered, the actual original difference of opinion is very small. All alike agree, I apprehend, that

classes must as a matter of fact admit of being aggregated in the ways just described. Moreover all alike agree that when we want to express 'X or Y, but not both,' we mark this distinctively. The only difference of opinion is in the treatment of 'X or Y or both':—shall we formally exclude double counting by our notation, or shall we assume as a general convention that it is not to be admitted?

Boole, as is well known, adopted the former plan, making all his alternatives mutually exclusive, and in the first edition of this work I followed his plan. I shall now adopt the other, or non-exclusive notation:—partly, I must admit, because the voting has gone this way, and in a matter of procedure there are reasons for not standing out against such a verdict; but more from a fuller recognition of the practical advantages of such a notation. The relative merits of the two methods will be more fully discussed hereafter, when it will be seen that each possesses some merits and some demerits; what turns the scale is the preponderant weight of brevity and symmetry in favour of the unexclusive notation. At present I will merely say that, as a rule and without intimation of the contrary, I shall express 'X or Y,' in its ordinary sense, by $X + Y$. I regard it as a somewhat looser mode of statement, but as possessing, amongst other advantages, that of very great economy. There are, however, certain logical operations,—to be presently described,—in which it is necessary to employ the other, or exclusive notation. I shall regard these as being exceptional cases, in which greater precision of language is required. When these occasions occur it will be necessary to take the precaution of seeing that our alternatives are,—either materially, or by formal expression,—cast in the exclusive phraseology. Due attention will be directed to the occasions thus indicated.

We must now take some notice of the attempts of popular language to express the above meanings. There is considerable laxity in our common vocabulary in this respect. Broadly speaking we employ two conjunctions, 'and' and 'or', for thus aggregating classes, to which 'with' and other words may be added, as occasionally used: these terms being occasionally synonymous in this reference, and all alike leaving it to be decided by the context whether or not they exclude the common part. Often there is no such part, the terms being known to be exclusive of each other; but if there be such, and the context does not make our meaning plain, we sometimes add a clause 'including both', or 'excluding both', or something to that effect, in order to remove all doubt. (The third of the recently mentioned cases, being comparatively exceptional, and hardly likely to occur except in some kind or other of numerical inference or application, may be left out of consideration.)

Thus, 'Lawyers are either barristers *or* solicitors', 'Lawyers consist of barristers *and* solicitors', 'The barristers, with the solicitors, comprise the lawyers' must all be taken as being equivalent statements. They all alike state that the class of Lawyers is made up of, co-extensive with, the two classes of 'barristers' and 'solicitors'. Whether a barrister can be a solicitor they do not give the slightest hint. Nor if such be the case do they unequivocally inform us whether these common members are to be included or not; though as inclusion is the far more usual case this would be very strongly assumed in the absence of any statement to the contrary.

There is, of course, a difference between the signification of these two particles. The word 'lawyers', when identified with 'barristers *and* solicitors', is taken somewhat more collectively, and when identified with 'barristers *or* solicitors' somewhat more distributively, as logicians say.

Hence when our subject is an individual, or a collective whole, a real distinction will be introduced by the use of one term rather than the other. Thus to say 'he is deceiver or deceived', is by no means the same thing as to say 'he is deceiver and deceived'. But the distinction here seems rather forced upon us by the necessity of the case than by the nature of the grouping of the two classes. The individual cannot, like a class, be split up into two parts[1]: accordingly his 'collective' reference to two classes forces us to conclude that one person at least must be common to both classes, that one deceiver must be deceived; whereas his 'distributive' reference to the two classes carries no such implication with it. This must rank of course amongst the many perplexities and intricacies of popular speech, but it does not seem at variance with the statement that regarded as mere class groupings, independent of particular applications, '*A* and *B*,' '*A* or *B*,' must in many cases be considered as equivalent.

In addition to these words, *and* and *or*, we have a variety of other words and phrases at our occasional service; such as, 'with', 'as well as', 'also', 'not excepting', and so forth—all of which serve the same purpose of aggregating class terms together so as to make them represent a single whole. But *how* they aggregate them, in respect of this matter of inclusiveness, must be a matter of interpretation and of reference to the context.

II. The next way in which we have to consider the mutual relation of classes is when one is *excluded* from another. It is often convenient to begin by taking account of some aggregate class, and then to set aside a portion of it and omit this from consideration. This process is the simple

[1] The differences resulting from this fact are so important that some writers (e.g. Schröder, and Voigt) have devoted a separate treatment to the case of individual propositions.

inverse of that discussed above, and some of the difficulties to which its expression has given rise spring from corresponding causes. Three cases may be noticed in order.

(1) The class to be excepted may be included within the other (or within the aggregate of the others, if the group consists of several). In this case the omission or exclusion is perfectly simple and intelligible.

(2) Again, if the classes are mutually exclusive, the omission of either from the other is equally simple, but unintelligible. The direction to omit the women from the men in a given assembly has no meaning. At least it will require some discussion about the interpretation of symbolic language in order to assign a meaning to it.

(3) But if the two classes are partly inclusive and partly exclusive of one another, we are landed in somewhat of a difficulty. What, for instance, would be meant by speaking of "all trespassers, omitting poachers"? If we interpreted this with the rigid stringency with which we treat algebraical symbols, we should have to begin by deducting the common part, viz. the poachers who trespass, from the trespassers; and should then be left with an unmanageable remainder, viz. the poachers who did not trespass, and who could not therefore be omitted.

Popular language is of course intended to provide for popular wants, and must be interpreted in accordance with them. Hence a laxity of usage is permitted in its case which could not be tolerated in the case of pure symbols. Accordingly when we meet with such a phrase as that in question, we take it for granted that any such uninterpretable remainder is to be disregarded, and we understand that 'trespassers, omitting poachers' is to be taken to signify 'trespassers, omitting such trespassers as poach'.

Here too we have a variety of phrases at command to

convey the desired meaning. The word most frequently used for this purpose is perhaps 'except', as when we say 'Lawyers, except Chancery barristers'. In this case we specify the subclass to be omitted, but sometimes we can express our meaning better by specifying the portion which is not to be omitted, as when we say 'Lawyers, provided they are Chancery barristers', which means that we are to except all who are not so. Besides these phrases we have a number of others at our choice, such as, 'omitting', 'excluding', 'but not', 'only if not', and so forth.

Before passing on to the next operation the reader may be reminded that this process of logical exception or subduction is not necessary for the completeness of our system. The process next to be considered can always be made to take its place; and indeed some high authorities reject it altogether. The nature of the substitution thus effected will be sufficiently treated in future chapters; but I may briefly indicate my reason for introducing this process here. It is not the object of Symbolic Logic, as here conceived and treated, to give only the most compendious scheme of formulæ to which our reasoning processes can be reduced. We want, at any rate as a commencement, to start with the current modes of thought, both those of common life and of the traditional logic, and to see how these can be expressed in symbols, and how they can then be developed and improved. Now it is quite certain that this process of 'excepting' or 'subducting' one class from another is generally recognized. And therefore we must take account of it here.

III. Our third logical operation in dealing with classes consists of selecting the common members from two or more overlapping classes. This is the statement of the process in respect of denotation,—the only side of terms, as previously

remarked, with which we are properly concerned. If however we choose to express the same thing in respect of connotation we should say that given two sets of attributes as distinctive of two different groups, we propose to confine ourselves to the attributes which occur in both groups. It is obvious that such a mode of class relation will as a rule result in a restriction or limitation of the number of things taken into account. This clearly must be the case; unless the two classes happen to be coincident, owing to our having really two names for the same group; or unless one class is entirely included in the other, in which case the combination of the two is equivalent to the neglect of the wider one. In this latter case we have limited the wider class, but have left the narrower one with its limits unaltered.

The way in which common language indicates this operation is very commonly by simple juxtaposition of the terms involved. This plan is always admissible when one term is a substantive and the other an adjective, and not unfrequently in other cases also; though as the logician, when he employs ordinary language, has to reckon with the grammarian, he naturally finds his freedom of expression hampered.

But though the readiest way of indicating the part common to two assigned classes is by simple juxtaposition of their respective names,—as when we say 'black men' to mark those individuals who are both black and men,—it is far from being the only way. Here, as elsewhere, the resources of language are only too copious and varied for the logician. We have quite a collection of popular phrases at disposal, differing from one another not so much in what they logically assert as in what they are conventionally understood to imply in addition. Sometimes this implication may be so strong that it is difficult to say where mere suggestion ceases and actual logical predication must be considered

to begin. Thus it comes about that such words as *and, but, accordingly,* together with a host of others, are all employed to express the process of selecting the common part in two overlapping classes. 'Zulus are savages and cunning', 'are savages but cunning', 'are cunning but savages', 'being savages are cunning', must all be understood as indicating exactly the same kind and degree of class limitation. They all alike assert that the class 'Zulu' is contained somewhere within the common part of the classes 'savage' and 'cunning'; or, if we prefer so to put it, that they possess the attributes distinctive of these two terms. Where they differ from one another is not in respect of what they assert but in respect of what they imply; the first containing no particular implication, the second implying the independent proposition that 'most savages are not cunning', the third that 'most cunning people are not savage', and the fourth that 'most, or all, savages are cunning'. But when we come to write them down in our logical language, we should be forced to reject all these implications, and to express them each alike in the form 'All Zulus are savage-cunning'.

Again; in common speech, adjectives are often ambiguous by not distinguishing whether they are predicative of a whole class, or merely selective out of it. When I read of "those who claim the dark-skinned Hindoo as a brother" I cannot be certain whether this phrase picks out certain Hindoos only, by this characteristic, or whether it is meant to inform us that all Hindoos possess the characteristic. As in so many other cases, we do not know what exactly is meant to be implied beyond what is directly stated.

The following simple sentence is one of many which might be offered to illustrate the combined redundancy

and deficiency of common speech in the expression of these class relations:—"The proper recipients of charity are those who are poor but honest, or sick and old, and those who are young if they be orphans". Here we are simply treating as an aggregate class the three classes describable as 'poor-honest', sick-old', and 'young-orphan', each of these three being the common part of two overlapping classes. For the purpose of expressing aggregation or addition we have used *and* and *or*; and for that of expressing the selection of a common part we have used *and*, *but*, and *if*.

We have now enumerated and briefly considered the three principal logical operations which are concerned with the mutual relations of our class terms. They are, I think, the only operations of the kind which would naturally present themselves to the mind. There is, it is true, a fourth operation which will have to be discussed in the next chapter, but it is not by any means an obvious one. Instead of being forced upon our notice like the above three, not only by logical necessity but by the requirements of daily speech and thought, it rather comes to us by way of our symbols. Its very existence may be said to be suggested by the wish to make our symbolic scheme complete and symmetrical. We will therefore set it aside for the present, until we have discussed the appropriate symbolic language for the three operations which are so familiar to us.

When we look about us in order to choose our symbols, those of elementary mathematics naturally offer themselves. Any such symbols that we shall need are very simple and almost universally familiar, so that it seems at any rate worth while to try if they will answer our purpose. Not of course that we propose to use them in the same sense as that primarily imposed upon them. On the contrary, the signification they will have to bear has been already

definitely settled for them in the foregoing discussion, and this distinctly varies from their natural or primary signification. Even where the analogy is closest, as in the aggregation of classes above described, we are not engaged in a process of *addition*. The process we have to perform may be best described, in familiar words, as that of throwing several compartments into one; but we have no thought of counting individuals or in any way adding up numbers. We do not 'add' together the English, French, Germans, and so forth, in order to make up the Europeans. Of course such a process of class aggregation may be made a *basis* of numerical calculation, as it may of many other operations with which it is in no way to be confounded. I cannot see that we are justified in any case in considering that there is more than an analogy, sometimes indeed a very close one, between these operations of Logic and those of Mathematics. Certainly the employment of the same symbols must not be construed into an admission that this is so.

I. To begin with the operation of aggregating two or more classes into one. This seems to be so naturally represented by the sign for addition, that one can hardly avoid writing down some such expression as $x+y+z$ to represent the class made up of x, y, and z[1]. No other formal or

[1] The systematic employment of this sign (+) for this particular purpose of class aggregation dates, I apprehend, from Boole. The sign itself had however been introduced into Logic long before. Thus Leibnitz (*Specimen demonstrandi*, Erdm. p. 94) writes $A+B$ to mark the addition of attributes or notions to form a more complex notion, in which he has been followed by others, *e.g.* Twesten (*Log.* p. 25) and Hamilton (*Log.* I. 80), but this is an interpretation in intension. The sign was also used by Ploucquet to mark the combination of assertions or premises, and by Maimon and Darjes to stand for affirmation in contrast with negation. Maimon however used another sign (|) in a case of exactly the same kind as that which we now mark by (+), as in $a+b \mid c \mid d$, for '*a* is either *b* or *c* or *d*' (*Versuch einer neuen Logik*, p. 69).

symbolic justification for it seems to be called for than these, —that the order of the terms thus connected is entirely indifferent, and that the aggregation of two groups is equivalent to the aggregation of the detailed classes which compose them; that is, we must accept the commutative and associative laws. Whatever sense we put upon the sign we must insist that $x + y$ and $y + x$ shall have precisely the same signification, and that $x + (y + z)$ shall be equivalent to $(x + y) + z$. It is so in Mathematics, and it must be so in Logic also if the symbol is to answer its purpose. That this condition is secured in Logic is obvious; for the order in which we group our terms is perfectly immaterial, and is recognized as being so in the common usage of *and* and *or*. Indeed common language has so thoroughly appreciated this fact that it does not seem to have prepared any pitfalls here against which the logician has to be on his guard. In whatever order we arrange the 'Jews, Turks, infidels, and heretics', we cannot extract any difference of signification out of the combined group.

II. The deduction, omission, or subtraction, (it will be seen that we can hardly help resorting at once to mathematical terms here) of one class from another is expressed with equal convenience by aid of the symbol $(-)$; so that $x - y$ will stand for the class that remains when x has had all the y's left out of it. The first point here that seems to call for symbolic justification is the ascertainment of the fact that the well known mathematical rule, about *minus* twice repeated producing *plus*, is secured in Logic. That is, we must ascertain that it is so, both in the mental processes we actually perform, and in the language which we use to describe them. And this may be established by a single suitable example. Thus if we describe the persons who may remain in a captured town as 'all the inhabitants except the

military, but omitting from these the wounded'; the wounded military are understood to be put back, by virtue of the two 'omissions', into the same position as the non-military;—provided, of course, that there are only two alternatives in the case. It is just a case of $x-(y-z)=x-y+z$. So the rule holds in Logic.

In using this symbol we must remember the condition necessarily implied in the performance of the operation which it represents. As was remarked, we cannot 'except' anything from that in which it was not included; so that $x-y$ certainly implies that y is a part of x. This is quite in accordance with the generalized use of the symbol in mathematics where it is always considered to mark the undoing of something which had been done before. Not of course that it must thus refer to the immediately preceding step, but that there must be some step or steps, in the group of which it makes a part, which it can be regarded as simply reversing. The expression $x-y+z$ will be satisfactory, provided either x or z is known to be inclusive of y, or if both combine together to include it[1].

This use of the negative sign must not be confounded with another use which has found favour from time to time, viz. that of employing it to designate negative terms[2].

[1] Leibnitz (*Specimen demonstrandi*) employed this sign, but his view of Logic being one of 'comprehension', $A-B$ meant, with him, the omission of the attribute B from the notion A; not the exception of B as a class from A as a class.

[2] Employed in this way, for instance, by Maimon and Darjes, and by a number of others. The earliest such use that I have seen is by Lambert: "In dieser Absicht liesse sich das Bindwörtchen *ist* durch das Zeichen (=), das *nicht* durch das arithmetische Verneinungszeichen (−) ausdrücken......Kein A ist M wäre $A=-M$" (Lambert's *Briefwechsel*, I. 396:—This is not Lambert's own notation). There is also a suggestion in this direction in Leibnitz's *De arte combinatoria* (Erdmann, p. 23), "Quemadmodum igitur duo sunt Algebraistarum et analyticorum prima signa + et −, ita duæ quasi copulæ *est* et *non est*."

The analogy here is very faint, and it is difficult not to believe that some of the logicians who have adopted this notation have been simply misled by the word 'negative'. Thus 'man' and 'not-man', taken together, should comprise 'all', i.e. the logical universe; but $+A$ and $-A$ taken together neutralize each other and result in 0. Moreover those who adopt this notation are very prone to yield to familiar association, and to think that they may transfer a term from one side of the proposition or equation to the other after changing its sign. Thus Garden says (*Elements of Logic*, p. 54) "So in Logic, by turning all the negatives into positives, and vice versâ, in both terms of a proposition, we propound the very same thing......If I say, All European nations, except the English, love to be abroad at night, and do not love coal fires, I state a proposition the terms of which, subject and predicate, are each composed of two terms, the one positive and the other negative......Now if I say, The English, and no other European nation, love coal fires and do not love being abroad at night, I manifestly assert the same thing, and do so by the process of changing all the signs on both sides......

All European nations $-$ English $=$ lovers of being abroad at night $-$ lovers of coal fires: or

All English $-$ European nations $=$ lovers of coal fires $-$ lovers of being abroad at night."

There seems to me much confusion here, arising from misuse of negative terms. The original assertion was one about 'Europeans excepting the English': we have no warrant therefore to draw, as he does, a conclusion about the English, for nothing was told us about them. Again, 'Europeans $-$ English' is understood correctly, as a case of subtraction: but 'English $-$ Europeans &c.' can only be understood as two assertions; an affirmation about the

English and a denial about the Europeans. Moreover he has confused an equation, which involves two predications, with a simple predication.

The question will naturally be asked whether we may really follow the mathematical analogy and transfer a term from one side of an equation to the other, with due change of sign. We are somewhat anticipating here, but the answer may best be given at once. Start with the case of a *sum* of terms, viz. with terms connected by the sign +. Here we can simply transfer a term if the alternatives are, either formally or materially, exclusive, but not otherwise. Thus, barristers and solicitors comprising the lawyers, the lawyers omitting the solicitors comprise the barristers ($x + y = z$; $\therefore z - x = y$). Similarly, if the 'shareholders' and 'bondholders who are not shareholders' compose the committee, then 'the committee, except the shareholders' will just consist of those who are bondholders only ($x + \bar{x}y = z$, $\therefore z - x = \bar{x}y$). But if the two classes overlap each other, we should then find that, in making the transfer, we were carrying this common part over: unless of course we are careful to make a corresponding correction. If, for instance, some members of the committee belong to both classes, we cannot say that the bondholders are determined by subducting the shareholders from the committee. That is, when $z = x + y$, and these overlap, we must deduct the exclusive part of x only, if we want the whole of y to be left ($z - x\bar{y} = y$). The power of free subtraction without the need of any such correction is one of the conveniences of the Boolian plan of notation.

As regards the same process when we start with a difference, instead of a sum, of terms, the case is simpler. A subtracted term can always be transferred, as a positive term, to the other side. That is, if $z = x - y$, we may always

conclude that $z+y=x$. The reason for the distinction is obvious, for in this case there is no question of overlapping to trouble us. We take it for granted that the original statement was accurately made, and that therefore y was included in x. Moreover, since z consists of the x's, after the y's have been excluded, it is plain that z and y must be mutually exclusive.

A very likely source of error may be pointed out here. By familiar rules of common logic we infer from 'z is x or y' that 'z which is not x is y'. The beginner may possibly be inclined, with Garden, to write such statements down in the form $z=x+y$, $z-x=y$; where it looks as if we had simply transferred x, by subtraction, and yet had obtained a correct result. The nature of the error thus committed will only become clear hereafter. But it may be pointed out at once that an equation is more than a predication, for it comprises *two* predications. What $z=x+y$ says is that z is *identical with* the sum of the x and y. But we have no right by common logic to say in consequence that 'z without x' is identical with y, which is what we declare by using the expression $z-x=y$. (As was shown above, the correct deduction would have been $z-x=\bar{x}y$; viz. that 'z without x' is identical with y that is not x.) The common logical inference of course did not commit us to this. It only concluded, in the predicational form,—using 'is' instead of 'is identical with',—that 'z without x', or which is not x, *is* y, which is quite correct.

III. The third logical operation, namely that of restriction to the common part of two assigned classes, may be represented by the sign of multiplication. That is, $x \times y$, or xy, will stand for the things which are both x and y.

The analogy here is by no means so close as in the preceding cases, but the justification of our symbolic usage

must still be regarded as resting on a simple question of fact: *i.e.* do we in the performance of the process in question, and in the verbal statement of it, act under the same laws of operation in each case, logical and mathematical alike? The answer is that we do so up to a certain point, but not beyond.

For instance the *commutative* law[1], as it is termed, viz. the indifference of the order of the symbols, prevails; xy and yx, say, English Protestants and Protestant English, being precisely the same class. So does the *distributive* law, that every additive element of the one multiplier shall combine with every such element of the other, that *e.g.* $x(y+z)=xy+xz$, and $(x+y)(z+w)=xz+xw+yz+yw$. Of course we have to make our way here through various grammatical obstacles; but allowing for them it is clearly true that we do thus combine and distribute our terms in ordinary speech, so that 'English and French soldiers and sailors', is the same as 'English soldiers, and English sailors, and French soldiers, and French sailors'. When however we say that this commutative law holds in common speech, we must remember that the idiosyncrasies of language insist upon exceptions here as elsewhere; and on the same ground, viz. the necessity of compressing a quantity of implication into our sentences, in addition to the direct assertions they contain. Thus when x implies y we always use the form yx, rather than xy. For instance one would not direct a legacy to those old servants "who are destitute and alive"; but to those "who are alive and destitute".

It will be equally obvious that if $x=y$ then $zx=zy$, that is, that we may 'multiply' equivalent terms by the

[1] The introduction of these technical terms appears to be of recent date. Hankel (*Vorlesungen*, p. 3) says that *distributive* and *commutative* (in this particular sense) were introduced by Servois in Gergonne's *Annales*, v. 93 (1814); and *associative* by Sir W. R. Hamilton.

same factor. For (as will be more distinctly insisted on presently) $x=y$ means that the classes of individuals which go by the names x and y are the same; consequently it comes to exactly the same thing whether we select the z's from them under their name of x, or their name of y. In each case alike we get the same result.

Where we depart from mathematical usage, or rather restrict the generality of its laws, is in the following respect. As a mathematical rule, xx (or x^2) is different from x. If x represent a number, then x^2 is greater or less than x, according as x is greater or less than 1; if x represent a line, then x^2 represents an area, and so forth. But in Logic x^2 must equal x, or rather any number of self multiplications must leave the significance of a term unaltered, *i.e.* $x=xxx$....... That this is so as an operation, is obvious, for the selection of the common part of two classes, when these classes happen to be the same, is reduced to the simple repetition of this one part. And that the same is true as regards the laws and usages of common speech must also be admitted. Or rather, it must be admitted under some exceptions; for there are signs of divergence towards the mathematical rule here, when we are dealing with *comparative* terms. Thus, even in English, 'great great' does not mean quite the same as 'great' alone; and in some languages, as Italian for instance, adjectival repetition is really almost like mathematical multiplication, increasing or diminishing the effect according as the term is in itself an augmentative or diminutive. If this conventional rule prevailed universally in respect of all, or even most terms, the logician would have to give way to the grammarian here, with the result of having to abandon Symbolic Logic, as a formal science distinct from Mathematics[1]. As it is, however, we feel at liberty to set aside

[1] Any one who has read Algebra will see the importance of this restriction in the processes of logical calculation. It reduces every logical

this 'comparative' usage as an occasional irregularity, and to lay it down universally that xx shall be considered the same as x.

It may be pointed out that this is not so much an infringement of mathematical laws as a special restriction of them. In fact there is one obvious case in which the same rule does hold in Mathematics, that is, when x stands for *unity*. In that case $xx = x$; as it may be said to do also when $x = 0$, or ∞ [1].

So far then seems plain. It must now be enquired what, on the suppositions thus made, will be the proper mode of representing certain important and limiting classes with which we shall often have to deal. How, for instance, shall we represent "all things":—or rather, to speak more correctly, how shall we represent the 'universe of discourse' with which we may happen to be concerned;—about the nature of which universe we shall have something more to say hereafter? It may at first occur to the reader that the most appropriate way of representing such a huge and miscellaneous assemblage as this would be by the sign for *infinity*, since Mathematics has such a sign at its disposal. We might resolve so to represent

equation to the *first degree*. Hence any equations with any number of terms are resolvable by the expenditure of sufficient time and trouble.

[1] There was a much freer employment of the sign for multiplication, than of that for addition, by the earlier logical symbolists. This was probably owing to two causes: partly, that here the comprehensive and extensive views come so much more nearly to the same thing; but also to the fact that popular language had already familiarized men with the juxtaposition of *words* to express the same results which the logician expresses by juxtaposition of *letters*. For instance Leibnitz uses the form *AB* for "*A* that is *B*", (Erdm. p. 102) and similar forms are of constant occurrence in Ploucquet's logical writings. But note how the latter describes them:—"Durch *mb* will ich keine Form der Multiplication sondern eine Associirung der Ideen verstanden haben", a very different conception this from that of the common part of the extension of *m* and *b*. (*Sammlung*, p. 254.)

it if we pleased[1]; but, if we did, we should soon find that we were acting inconsistently with the plan we have just adopted of indicating combined attributes or overlapping classes by the sign of multiplication. When once that plan is adopted we are almost bound in consistency to make the symbol for unity, or 1. stand for the universe. That this is so can very readily be shown. For what does our x stand for? It comprises *all things* which are x. In other words, one way of describing and getting at x is to say that we combine, or take the common part of, the universe and x, just as 'English men' may be got at by taking the common part of the classes which are English and which are men. We do not say that this is the only way of getting at it, but it is one way, and therefore our results must be consistent with so reaching it. Now the only known symbol which when combined, in accordance with the rules for multiplication, with any symbol x, will always still give us x, is unity. Hence the one universal class must be represented by the sign for unity, so that $1 \times x$ shall always equal x.

The interpretation of the commutative law in the case of $x \times 1 = 1 \times x$ deserves a moment's attention. If we regard the expression as a process of selection rather than as the indication of a result (which is not however the best way of regarding it) we should say that $x \times 1$ may be described as the process of selecting from the whole universe those things which are x. So $1 \times x$ will be the selection from x of all those things which belong to the universe, which of course leaves x unaffected.

On the same principles a non-existent class, or, as we shall often find it convenient to describe it, an empty compartment, will be fittingly represented by the sign for nothing, or 0.

[1] It has indeed been so represented by more than one symbolist. (See on ch. IX.)

This looks natural and plausible enough; but it is well to point out that it follows strictly from what has hitherto been laid down. For if xy stands for the things which are both x and y, then $0x$, or 0, will stand for those which are both nothing and x; in other words for the non-existent, or for any empty class.

The sign of subtraction supplies us in the same way with a suitable expression for the logical contradictory of any class x, viz. for the class 'not-x'. For since not-x comprises everything which the universe contains except what is x, we should write it symbolically[1], $1-x$. Now combine x and $1-x$, and what do we thus indicate? The class of things which are at the same time x and not-x. Of course this is bound to equal *nothing*; *i.e.* $x(1-x)=0$. This supplies a link of consistency with some of our previous assumptions. For 'multiply out' $x(1-x)$, as in Algebra, and we have $x-x^2$ or $x-xx$; which as just remarked, is to be equal to zero, and this necessitates that x and xx, or x^2, shall be identical with each other, which we already know that they must be.

One or two examples may be added in illustration of the employment of our symbols, and of the restrictions to which they are subject. Suppose we have the expression 'English and French Poets and Orators'. Following the indications already given we might write it down symbolically $(a+b)(c+d)$, thus combining the four terms by the signs respectively of addition and of multiplication. Multiplied out, the latter becomes $ac+ad+bc+bd$. Now the original verbal statement, when fully expressed in its details, is equivalent to English poets and English orators and French poets and French orators; that is, the symbolic process of multiplication gives exactly the same combinations of terms that are ob-

[1] For the sake of brevity we shall commonly write $\bar{x}$ for $1-x$, or not-x. It is an abbreviation introduced and frequently employed by Boole.

tained in common discourse; the two corresponding step for step, as of course they are bound to do.

In this case the condition of what might be called punctiliously accurate expression previously alluded to, about the mutually exclusive character of our terms, does not obviously make itself perceptible, but that is because correction is tacitly made where required, when the symbols are translated into words. Since a and b, standing for French and English, are already exclusive, no correction is required. But c and d, standing for poets and orators, *do* overlap, and therefore $c+d$ does *symbolically* count this common part twice. We make therefore the due correction at this stage. But working symbolically we come upon the same repetition again in the final form $bc+bd$. Here again we make a correction of the strict symbolic import of this expression, in order to secure what we know to be meant; accordingly we conclude that the orator-poets are only to be reckoned once.

In order not to multiply examples needlessly we will take one more which involves the sign of subtraction, and see how that answers when worked out. Take, for instance, $(a-b)$ $(c-d)$ and see what it might stand for in common language, both in its present shape, and as it would become when multiplied out in detail. The following would be a fairly corresponding verbal expression:—'Barristers (excepting foreigners) who are graduates, but not of Dublin'. What does this mean? It seems quite clear that all whom we can possibly intend to except from the barristers are foreign barristers, not foreigners in general, and that all whom we except from the graduates are Dublin graduates. In other words, this process of exception or subtraction always presupposes that the class excepted is a part (formally or materially) of the class from which it is excepted. We are therefore warranted in writing our symbols $(a-ab)$ $(c-cd)$ or $a(1-b)c(1-d)$, and in re-

garding both this and the verbal statement as being compounded of four terms, two of them positive and two of them negative; and which might be expressed still more briefly as $a\bar{b}c\bar{d}$.

But the question was not so much whether the two expressions corresponded as they thus stood, but whether the result of multiplying out the symbols could be translated step by step in its details and shown to correspond to the results of ordinary thought. Those symbols, when treated by the ordinary rules, yield the result, $ac - acd - abc + abcd$. Is there anything in the statement about the barristers corresponding to all this? Certainly there is, for these symbols as they thus stand in detail, are literally translatable into the words 'Graduate barristers, omitting Dublin graduate barristers, omitting also foreign graduate barristers, but adding on Foreign Dublin graduate barristers'. If we go through this we shall see that it expresses exactly the contemplated class, but with one important proviso. This proviso is that the expression 'omitting' here is not to be taken in its popular signification, by which its application is tacitly understood to be limited to the range of the term which precedes it, but in its strict symbolic signification in accordance with which we may *omit too much*, and therefore find it necessary to add on a term to correct this excess. This, of course, is the real meaning of the final term $abcd$. Common language left to itself, would have compendiously expressed the details in the form $ac - acd - abc$, that is, would have put it 'Graduate barristers, omitting those of Dublin and the foreign ones'. The accurate language of symbols requires us to insert a final term which common language had rejected for the sake of brevity.

The above examples are fairly illustrative, and will serve to explain some of the main points of our system of Symbolic

Logic. We see that we may employ the sign of addition (+) for the aggregation of classes. We may employ the sign of subtraction (−) for the exception of one class from another, provided the excepted class is included in the other (common language appears so uniformly to take this for granted that we may consider it as really intending this limitation, though its terms do not formally imply it). And we may employ the sign of multiplication (×) for the results of selecting the common members of two classes. (This process is so simple that common language does not seem to have become loose or inaccurate here, though it puts an occasional intensifying effect upon the act of repetition of comparative terms; a perfectly natural intensification, but one which we find ourselves compelled to reject.)

To one point, which has been already noticed, attention must be very persistently directed, as any vagueness of apprehension here will be fatal to the proper understanding of symbolic reasoning. It was seen in the last example that, in translating popular language into symbols, we felt justified, or rather bound, to ask ourselves what exactly that language must be understood to mean, instead of just proceeding to put it into symbols as it stood. We may say in fact that we were resolved to give an *idiomatic* translation and not to resort to a bald and literal substitution of terms. 'Lawyers excepting foreigners' was accordingly interpreted, on this ground, to refer to what was left after foreign lawyers, not foreigners in general, were left out of account. Accordingly we translated it into our symbols as $a-ab$, and not $a-b$, for we knew that the former was what we meant. But in the converse case, that is when we were translating back from the language of symbols into that of common life, we had no right to do anything of this kind. Symbols have no tacit limitations or con-

ventional interpretations, other than that which has been strictly and originally assigned to them. Accordingly if I meet the symbolic expression $a - b$, where a stands for lawyers and b for foreigners, I have no right to regard this as a popular lax expression, in which only a part of b instead of the whole is to be subducted. I must take it as signifying (whatever that may mean) the subduction of *all* foreigners from the lawyers; and I must look out for, and, if the processes have been correctly carried out, shall find, the introduction of another term which will set matters symbolically straight, by just neutralizing the surplus subduction; common language on the other hand sets matters straight by interpreting the subduction in its own sense, and thus neglecting the required additional term.

It comes therefore to this. We may translate *into* the language of symbols almost as we please, for we are the only ultimate judges of what we really mean to signify by our words. But having once done so, and laid down rules for working with our symbols, our further control of their meaning ceases. We must translate out of these symbols into the language of common life in exact accordance with their assigned meaning. Otherwise we shall find that one correction and modification after another will be called for.

It will now be seen how it is that the process of subtraction, with the corresponding symbolic sign, can be dispensed with. If y be excepted from x it must be a part of x, and may therefore be written xy ('Graduates except Dublin men' are the same as 'Graduates except Dublin graduates') so that we have $x - xy$, or $x(1-y)$. But this, as we have just explained, is the same as to 'multiply' x by $1-y$, or by not-y. In other words, the legitimate exception of y from x is the same thing, in respect of the result, as taking the common part of x and not-y. 'The clergy, except the

teetotalers', means, when the exception is duly interpreted, the class common to those of 'the clergy' and 'the non-teetotalers'. In other words, speaking generally, the employment of negative terms connected by the multiplication sign will take the place of that of positive terms connected by the subtractive sign. It was from not realizing this that Mr Garden's blunder, noticed above, arose. Starting with "(European) *minus* (English)" as subject, the proper mode of changing the sign would have been to substitute "(European) × (not-English)"; not "(English) *minus* (European)".

The rules, therefore, for logical subtraction can certainly be dispensed with. The notion, however, of this process of exception, in a sense analogous to (but not identical with) that of arithmetical subtraction is far too familiar to popular thought for us to be able to ignore its existence, though we shall not very often have occasion to employ it in practice.

Whilst on the subject of 'multiplication' two very useful formulæ may be noticed. Most of the cases which occur in practice are so simple that the beginner will have no difficulty in carrying them out. There are, however, some cases of such frequent occurrence that it is well worth while to keep in mind a few simple rules for the abbreviation of a process which is often apt to prove tedious.

(1) $(A+x)(B+x)=AB+x$. There are of course four terms in the full product, but two of these may be omitted on the non-exclusive plan of addition, (viz. Ax and Bx), because they are included in x. As A, B, and x, may be classes of any degree of logical complexity, this effects great simplicity. All that we have to do, when multiplying two such classes together, is to take note of any common part they contain. We then multiply the remainders only, and add on this common part.

(2) $(A+x)(B+\bar{x})=A\bar{x}+Bx$. Of the four resultant

terms, one (viz. $x\bar{x}$) simply disappears; and another (AB) is omitted because part of it is included in $A\bar{x}$ and the rest in Bx. It need hardly be said that here, as above, A, B, and x, may be logical classes of any degree of complexity; so that considerable simplification may be thus effected.

The former of these rules is given, as a special method of simplification, by Boole (p. 132). The latter, with the full significance of these abbreviations, is due to Professor Peirce.

IV. The only remaining mathematical symbol to which we will at present direct attention, and almost the only other one which we shall have occasion to adopt, is the sign ($=$) of equality. Here too we must not trust too much to acquired associations. What this symbol generally means in mathematics is identity (or indistinguishable similarity) in respect of some one characteristic only in the various things which it connects together; this characteristic being in most cases the number of *units* involved. These may be units of space, or time, or mass, or acceleration, or what not. Thus in the case of a falling body, $v^2 = 2fs$ means that the square of the number of units in the velocity is the same as twice the product of the number of units of force and space. But in logic this is not so. The sign of equality here indicates absolute identity in all respects, except nomenclature, of two logical classes. The identity indeed is so complete, that all that needs pointing out is how we can talk of two distinguishable classes in such a case. The answer of course is that a 'class' is merely our way of grouping or *outlining* things, and is indicated and retained by the imposition of names. The very same individuals therefore may belong to more than one class; in other words, may have more than one single name or combination of names assignable to them. It is in this sense only that

we can talk of the identity of all the members of two or more classes. Such identity is what we mark by the sign of equality (=).

Here, as in the case of our other symbols, we must turn to see by what sort of contrivances popular language succeeds in conveying its meaning. As usual it is vague, but shows both defect and redundancy, the signification being controlled by innumerable conventions and implications. One way of expressing the identity in question is by mere predication. If in answer to the question, What are triangles? I reply, 'Triangles are plane figures included by three straight lines' I clearly mean that every thing referred to by the one name is also referred to by the other; that is, that the individuals in these classes are identical. This is however rather a lax rendering, for the copula 'is' (or 'are'), by itself, properly implies nothing more than predication. The stricter form for expressing this identity of classes is by some such words as "consists of", or "comprises"; and even these may sometimes need the addition of some other clause such as "and includes no others" in order to remove all ambiguity, and to make it plain that we contemplate a case of identity of classes, and not merely of the inclusion of one class within another. Again the word "means" implies this identity, and implies it somewhat strictly. It does not properly speaking *state* it; for the two things which we thus connect in our proposition are not so much two classes of individuals, as the significance of some word on the one hand and the things it refers to on the other. But it certainly carries with it this complete identity of classes. When I say that 'ghost' means a disembodied human spirit, I imply that the class of things, real or imaginary, referred to by one name is identical with that referred to by the other name.

Symbolic Logic—at any rate as here interpreted,—may

therefore fairly be said to take an ultra-nominalistic view of this subject. The expression $x = y$ simply tells us that the class of things of which x is a name is composed of the very same members as that of which y is a name. And $xy = a + b$ tells us that the restricted class of things, of which both the names x and y simultaneously hold, is identical with the extended class of things covered by the two names a and b.

CHAPTER III.

SYMBOLS OF OPERATIONS (CONTINUED).

THE INVERSE OPERATION, $\dfrac{X}{Y}$; OR SYMBOLS OF DIVISION.

WE have now seen our way clearly enough to the performance, and symbolic expression, of three distinct logical operations upon classes, viz. :—

1. A direct operation closely analogous to the addition of ordinary arithmetic and algebra, and suitably symbolized by the familiar sign (+).

2. The inverse operation to the above, and therefore closely analogous to subtraction, and which may be suitably symbolized by the sign (−). We pointed out that this process, though not indispensable symbolically, was quite recognized in popular thought and speech.

3. A direct operation very remotely analogous to multiplication[1], and which we have seen could (with one restriction in respect of usage) be expressed by the usual signs for that process, viz. (×) or simple juxtaposition of the terms.

[1] As Prof. Schröder points out, the closest arithmetical analogy with logical addition (in the non-exclusive sense) is found in the process of obtaining a smallest common multiple; and with logical multiplication in that of obtaining a greatest common divisor, of two quantities. (*Vorl.* I. 253.)

The question therefore at once suggests itself whether there may not be a fourth operation which shall be the inverse of the third, as the second was of the first. This suggestion, it must be admitted, comes to us rather by way of the symbols than by way of the actual logical process itself. In this respect it is unlike the first three. As regards them we saw that it was quite impossible to think and speak about classes of objects without having these three operations forced upon our notice: in fact, they are so familiar to us that our trouble arises rather from redundancy, than from lack, of phrases to express them. The symbols are therefore an afterthought to express results and operations to which we are already accustomed, though their introduction is a powerful means of economizing time and thought.

But at this stage we take a different course. We might conceive the symbols conveying the following hint to us: Look out and satisfy yourselves on logical grounds whether there be not an inverse operation to the above. If you can ascertain its existence, then there is one of our number at your service appropriate to express it. In fact, having chosen one of us to represent your logical analogue to multiplication, there is another which you are bound in consistency to employ as representative of your logical analogue to its inverse, division,—supposing such an operation to exist.

A few words of reminder as to the nature of an inverse process[1] may not be out of place here, especially as the

[1] The reader who can do so with profit is strongly recommended to consult the works of some of the more philosophical mathematicians for a discussion of the distinction in question. It is the symbolic language of mathematics only which has yet proved sufficiently accurate and comprehensive to demand familiarity with this conception of an inverse process. See, for instance, Boole's *Differential Equations*, ch. XVI. There is an admirable discussion of some of the general charac-

application of this term, just above, to subtraction, may suggest too simple a notion of its general nature. The relation of Interest to Discount, as theoretically treated in elementary works on arithmetic, will furnish a familiar instance. Take a sum of £100, and put interest upon it at 5 p.c. for a year; and call this the direct process. The result will of course be £105. But now suppose that we had been asked instead to find a sum such that, when a year's interest was added on to it, we should obtain £100, we should have been called upon to perform the inverse to the above. The result of course is £95. 4*s*. 9*d*. In this case it will be observed that the inverse process is just as definite as the direct, that is, there is but one sum possible such that, with the interest added on, it shall yield the given amount. In mere arithmetic this is generally so. To subtract 5 from 20, or in other words to find a sum that, with 5 added on to it, shall become 20, can only yield the result 15. So it is with multiplication and division, and with the calculation of logarithms, powers, and most other simple functions.

Very often, however, what are called inverse operations are indefinite. Instead of there being only one starting point such that the performance of the direct process would carry us from it to the desired result, there may be a plurality of such starting points, or even an indefinite number of them. Thus in Trigonometry the calculation of the sine of a given angle is definite:—*e.g.* $\sin 45^\circ = \frac{1}{\sqrt{2}}$. But the calculation of the angle whose sine is given is indefinite:—thus $\sin 135^\circ$ is also $\frac{1}{\sqrt{2}}$, or generally, $\sin^{-1} \frac{1}{\sqrt{2}} = (2n + \frac{1}{2})\pi \pm \frac{\pi}{4}$.

teristics of the symbolic language of Mathematics in De Morgan's *Double Algebra.* Also in Hankel's *Vorlesungen über die complexen Zahlen.*

It is to this class of indefinite inverse operations that the logical process which we indicate by the sign of division will be found to belong.

In strictness these terms *direct* and *inverse* are purely relative, and of indifferent application; that is, whichever of a suitably related pair of processes we may choose to call direct, we may call the other its inverse. But the fact that the one process is very frequently definite whilst the other is indefinite, makes such an important practical distinction between them, that we are in the habit of applying the term absolutely to the latter; and we speak of *the* inverse process, whilst we call the definite one the direct process[1]. It is in this sense that we shall call the logical process which will be indicated by the sign of division, an inverse operation.

It should be remarked that the inverse rule merely indicates a result, giving at the same time a test by which that result is to be verified, but that it does not describe in any way the process by which this is to be obtained. It does not even assure us that there is any known method of obtaining it. The rule says in effect: Find something, which, when operated on in a certain way, will yield a certain result; but it does not tell us *how* we are to set about finding that something. As Boole says (*Differential Equations*, p. 377), "It is the office of the inverse symbol to propose a question,

[1] Even where both processes are definite, one of them may be much more difficult of performance than the other, or even impracticable. Thus it is easy enough to calculate the fifth power of a given number, but we have not any simple and complete arithmetical rule for calculating a fifth root: *i.e.* otherwise than by continued approximation. There seems some disposition to apply the term 'inverse' in an absolute sense to cases where no rule of operation can be given, and where in consequence we can only test a proposed result by means of the corresponding or direct process.

not to describe an operation. It is, in its primary meaning, interrogative, not directive[1]."

To find then the inverse of the process indicated by xy, that is of the direct multiplicative operation of x upon y, or y upon x, we must recall what that operation is. It is, generally speaking, one of *restriction*[2]; it is the act of confining the attention to the common members of the two classes x and y; so that to operate upon x by means of y is to restrict the class x by exacting the condition of being y also. What then will be the inverse of this, represented symbolically by $\frac{x}{y}$? Not, as some might be tempted to reply at once, the mere *taking-off* of this y-restriction from x, but rather the finding of a class such that, *when the y-restriction is imposed upon it*, it shall be brought down exactly to x.

What then must be the description of the desired class, and can it be determined by ordinary logical considerations? Certainly it can, provided we set to work methodically by examining all its possible components in order. When we proceed to do this, we see, as a first requisite, that such a class must certainly contain the whole of x. As regards what is not-x it certainly cannot contain the y part, (or $y\bar{x}$), for, if it did, this part would not fulfil the condition of reducing to x on imposition of y. As regards what is

[1] It seems surprising that one who had so clearly stated the nature of an inverse operation in mathematics should never have explicitly proposed any corresponding explanation in Logic.

[2] We call it restriction, because this is the usual result; but in special cases the class thus determined will clearly not be narrowed. For instance, if x and y are identical in extent, then xy has the same range as either x or y. (If 'all x is all y' then xy is no more restricted than x or y simply.) Or again, if x is already included in y, then xy will be no narrower than x, but will coincide with it; so that it is only y that is restricted by the process. As all men are lung-breathers, the lung-breathing men are no narrower a class than the men in general.

neither x nor y, it may take as much or as little as it pleases, that is, a perfectly indefinite portion; for any such portion will satisfy the condition, by disappearing on combination with y and leaving us x only. The full description therefore of the desired class is given by saying that it comprises the whole of x, and a quite uncertain part of $\bar{x}\bar{y}$, namely of what is neither x nor y. If we like to put a peculiar symbol (v) to represent perfect uncertainty of range of application, we should write down the class symbolically as $x + v\bar{x}\bar{y}$.

The test of the correctness of this result is found, as in the case of other inverse operations, by simply performing the direct process upon it, and seeing whether we are thus led back to our original starting point. Thus if we multiply this expression, $x+v\bar{x}\bar{y}$, by y, it must yield x. It is true that at first sight we seem to get a different result, viz. xy instead of x; but the difference is apparent only, inasmuch as xy and x are in this case the same.

The point here touched upon is very important in symbolical reasoning, though its full significance will only come out in a future chapter. The fact is simply this: that the very stating of such a problem presupposes a condition as to the mutual relations of x and y. To ask for a class such that on restriction by y it shall just reduce to x necessarily implies that all x shall be y, as otherwise the process could not be performed. If there were any part of x that is not y this part would of course disappear on restriction by y; that is, a part of x would have disappeared from our result, whereas the question demanded that we should be left with the whole of x on our hands and nothing else. Accordingly when we are asked to perform the inverse process indicated by $\frac{x}{y}$ it is necessarily assumed that 'all x is y,'

or that x and xy are the same; as are consequently $\bar{y}$ and $\bar{x}\bar{y}$. The symbolic expression of the desired class may therefore be written down indifferently $x + v\bar{y}$ or $xy + v\bar{x}\bar{y}$. The 'proof' of the result is that either of these expressions when multiplied by y reduces to xy, that is, to x, as the problem demands.

It will be observed therefore that we may speak of the expression $\frac{x}{y}$ as standing for a certain logical class. It is particularly necessary to call attention to this. I regard it as an essential part of our scheme that this symbol, like all others which are admissible into Logic, represents a class or class-group of some kind or other. The class may be complicated in its determination, that is, it may require a number of terms to assign it; and it may, in certain directions, be indefinite in regard to its limits, as in the above example, but it is never anything else than a true logical class or conceivably assignable group of individuals.

The reason for laying such stress upon this point is that the contrary view is very widely assumed. It is generally maintained that the sign of division is uninterpretable[1]. Of

[1] I admitted, to some extent, this uninterpretability myself (in an article in *Mind* for October 1876). At least, I could not then see my way with certainty to any other than that *interrogative* explanation which is discussed on the next page, and therefore assumed that we must admit the fraction $\frac{x}{y}$ as being an uninterpretable stage in the process of deduction. But that essay was little more than a brief account of Boole's actual method; and not being then concerned with the re-investigation of his principles I was content in various directions to take them temporarily for granted. It seems clear that Boole himself so regarded it, for he says expressly (p. 69) "the chain of demonstration conducting us through intermediate steps which are not interpretable to a final result which is interpret-

course if we suppose that the sign stood for nothing but ordinary division, so that $\frac{x}{y}$ meant 'divide x by y'; then it must clearly be admitted that such a sign would be meaningless in Logic[1]. But then, for that matter, so would xy if it meant 'multiply x by y'; for Logic makes no direct use of either of these operations. But if we frame our demand thus: Perform the inverse of that logical process which you have elected to indicate by the sign of multiplication, then the sign lends itself readily enough to a comparatively simple interpretation. The required interpretation is merely this:—the expression $\frac{x}{y}$ stands for a class, viz. for the most general class which will, on imposition of the restriction denoted by y, just curtail itself to x. But to this expression we must remember to attach the condition, that this presupposes that 'all x is y', as otherwise no such class as that which it is desired to determine could exist.

The above is one way of approaching these inverse processes, and it seems the most immediate way. But as the conception is decidedly unfamiliar, it will be well to look at it from more than one point of view. It has been shown that when we try to solve the inverse problem we find that it necessarily presupposes a condition of relation

able." And, further on, "The employment of the uninterpretable symbol $\sqrt{-1}$ in the intermediate processes of Trigonometry furnishes an illustration of what has been said." (I need hardly say that I do not myself accept the uninterpretability of the symbol $\sqrt{-1}$.)

[1] The reader of 1894 may possibly be surprized at this statement, purposely left as it was made in 1880. If he refers to any of the works of Jevons,—then and for some years before, the only widely known expositor of anything that could be called Symbolic Logic,—he will find repeated declarations that any such fractional forms are absolutely uninterpretable in Logic.

between x and y. Suppose we start with the explicit assertion of a condition, by asking the question, 'If the z which is y is the same as x, what is z in general'? we shall find that we are doing exactly the same thing as asking for the inverse of xy. For express this condition symbolically and it stands $zy = x$. Now suppose that we proceed to treat our symbols in reliance upon the belief that the process corresponding to division could be performed, or rather, that when indicated it could be submitted to explanation. We should go on to conclude that $z = \frac{x}{y}$; in other words, we should be led on to use and interpret that symbolic form which we have just been discussing immediately and at first hand.

It is in this manner, it may be remarked, that these fractional forms will almost always originate in practice. We have some term given to us qualified or limited by a condition, and we are asked to determine all that we can about it without such condition. But whether the process be called for directly, as before, or, in consequence of a stated condition, as here, it is always essentially the same. We have assigned to us a restricted class, and a restricting class, and we are called upon to determine with the utmost generality the class which on combination with the latter will reduce precisely to the former.

A simple concrete example will serve to make this plain. For instance, given that Peers are the same as English Aristocrats, what are aristocrats in general?—the answer to be given of course in terms of *peer* and *English*. Aristocrats, when restricted by the class *English*, become *peers*; what can we say about them when subject to no such restriction? A little reflection would lead to one or two exclusions. They cannot be non-English peers, for it is definitely implied that there can be none such; nor again, from the meaning of the

terms, can they be non-aristocratic English. But they must certainly include all the peers. Hence they can only be described generally as including 'all peers, together with a perfectly indeterminate number of what are not English nor consequently peers.' That is, symbolically, if $yz = x$, then $z = x +$ possibly any portion of what is neither x nor y. If, as before, we put v as indicative of this kind and degree of indeterminateness, we should write it $z = x + v\bar{x}\bar{y}$. (It is also clear here that $x = xy$; that is, that 'peers' are the same as 'English peers', so that we might equally write it $z = xy + v\bar{x}\bar{y}$, as already pointed out.)

We have gone thus minutely into the discussion of this question, because the requisite conceptions are decidedly unfamiliar even to logicians[1]; but there can surely be no mental exercise more beneficial than that of generalizing to the utmost the processes we perform, and realizing clearly such distinctions as that between direct and inverse operations. Of course if the inverse process of class-expansion were as important in common thought and speech as the direct process of class-restriction is, it would become desirable to adopt some simple verbal expression for indicating it. Just as we now write 'English aristocrat' as a brief expression for the restricted class common to the two classes, so we might write $\frac{\text{Peer}}{\text{English}}$ as a brief expression for the expanded class obtained by the above-mentioned process, namely for the class 'peers, together with (possibly) any

[1] A word of friendly remonstrance may be offered to those logicians who happen to be anti-mathematicians. It is of no use rebelling against the introduction of these processes on the ground that they are not logical. Whether we employ symbols or not, the processes themselves must be introduced; and indeed are so, though in a very rudimentary stage, (as will be shown in a future chapter) even in the common Logic. What is wanted is that they should be discussed in their full generality.

portion of what are neither peers nor English'. We need be no more suspected of wanting to divide the peers by the English in this case than of wanting to multiply the English by the aristocrats in the other. We should be using nothing but convenient linguistic forms for operations which we had occasion to perform. As a matter of fact, since we hardly ever do have to perform the latter process, except as a deliberate logical exercise, no necessity for the introduction of such a verbal device has arisen. But we must not forget that our symbols are simply a substitute for our terms or words, and therefore any scheme of thought and language which can find a rational use for the symbolic form $\frac{x}{y}$, might conceivably find it desirable to introduce such a form as $\frac{\text{Peer}}{\text{English}}$ into ordinary language, in order to express its current wants.

The mathematician may be interested in noticing that the logical process indicated by $\frac{x}{y}$ has, in one respect, more analogy with integration than with division. Division, though an inverse operation in respect to multiplication, is perfectly definite. The direction to find a quantity which multiplied by 4 shall yield 20, is just as definite as the direction to multiply 4 by 5. Only one answer is yielded in either case. But the direction to find a quantity which when differentiated with respect to x shall yield, say, a, is indefinite. It can only be described in the form $ax+c$, where c may be anything not involving x; ($\int adx = ax + c$). We are bound, that is, to allow for the possible existence of an unknown or indeterminate surplus term which will vanish on the application of the process of differentiation.

The inverse logical process in question has this one

characteristic in common with the above, of yielding an unknown or indeterminate term which disappears on application of its corresponding direct process, and for that reason has to be allowed for and inserted. We say that $\frac{x}{y} = xy + v\bar{x}\bar{y}$, where the term $v\bar{x}\bar{y}$ corresponds to the constant c in the integration. Whatever c may be, $\frac{d}{dx}(ax + c) = a$; and so whatever $v\bar{x}\bar{y}$ may be, $y \times (xy + v\bar{x}\bar{y}) = xy$. Consequently just as we write $\int adx = ax + c$, so we must write $\frac{x}{y} = xy + v\bar{x}\bar{y}$, or in its simpler form $x + v.\bar{y}$.

There is another explanation which has been offered for this sign of division, and which deserves notice here, as does also the corresponding explanation for the sign of multiplication. It is that the former sign represents the process of logical *abstraction* and the latter that of logical *determination*[1]. Against the general propriety of this view no objection can be raised, abstraction and determination being of course strictly logical processes, and therefore falling

[1] This view has been held by several recent logicians, *e.g.* Schröder (*Operationskreis*, p. 2) and his reviewer, Günther (*Vierteljahrsschrift für wiss. Phil.* 1879, p. 111).

The explanation in question however is much older than is commonly supposed, having been suggested by Lambert and by G. J. Holland, more than a century ago, in the course of the discussion which they and Ploucquet carried on together about the principles of Symbolic Logic. Thus Holland says, "If from the concept A the partial concept b is abstracted let the resultant concept be termed R. Then we have $R = \frac{A}{b} = ac$ (say). But one is no more dividing here than one is multiplying in Composition" (J. H. Lambert's *Deutscher gelehrter Briefwechsel*, Vol. I. p. 261. 1768). Lambert himself has employed the same notation (*Nov. Act. Erud.* 1765, p. 454:—for some account of which scheme in the particular case of negative propositions see Chap. XX.). Lambert has also remarked that Determinations (*Bestimmungen*) may be indicated by the sign of multiplication. (Ploucquet's *Sammlung*, p. 223.)

well within the limits of any logical system. This explanation is also on the right track, but it does not seem to me quite to hit the mark.

My main objection to such an explanation is that it rests too much upon the connotative force of the terms we use instead of appealing solely to their denotation. (The grounds for thus defining our terms entirely in regard to their class limits, instead of in regard to their 'meaning', or the attributes they imply, is discussed more fully in the next chapter.) It is quite true that the product xy may frequently be described as the process of combining the attributes connoted respectively by x and y and so obtaining a limited or more 'determined' class. But it seems decidedly better to say simply that this represents the members common to these two classes x and y, without reference to their attributes; both because some terms may have nothing but denotation, and because where they have connotation also, this latter element is far less easily ascertained.

Now though it is true that Abstraction represents a process the reverse of class restriction, and so far corresponds to our 'analogue of Division', yet it seems to do so with such limitations, and to presuppose such conditions, as to unfit it to be regarded as general enough for our purpose. We cannot for instance abstract an attribute from a term unless such attribute is distinctly implied in the meaning of the term. We can abstract, say, rationality from man, because 'man' means 'rational animal'; but there seems no traditional warrant for saying that we can abstract the property of 'having two eyes', or any other property not included in the connotation. When we use the form $\frac{x}{y}$ we assume, it is true, that the class x is included in y, (or that $x = xy$), but this is a far more definite and determinable fact than to decide

whether certain attributes of y are or not included in the connotation of x. To decide that 'all x is y' is often simple enough, but every one knows the delicacy of deciding what is implied in the 'meaning' of x and of y. Similar remarks apply to the logical process of Determination. Indeed it is quite a question whether, with Professor Wundt (*Log.* p. 225), we ought not, in consistency, to go the length of rejecting the Commutative law when we take this view of the process. The determining and the determined are not, so to say, symmetrical; the process of determining A by B is not always identical with determining B by A. I have therefore generally preferred to speak of 'class restriction' which is less open to such objection.

Again, Abstraction is limited in another way which unfits it for our purpose. It is regarded as yielding a definite class, whereas we require one which shall be to a certain extent and in a certain direction, indefinite. Thus the abstraction of 'rationality' from 'man' would be considered to give 'animal', because man is defined as rational animal. But on our system $\frac{\text{man}}{\text{rational}}$ would represent 'rational man *plus* an uncertain portion of what was not rational nor man'[1].

All that I feel able to say therefore is that if we had to select recognized logical terms to express the analogues of multiplication and division, Determination and Abstraction would be the best for the purpose. But as things are, these terms are too much connected with the connotative force of

[1] Let z = man, x = rational, y = animal. Then $z = xy$, $\therefore y$, or 'animal', $= \frac{z}{x}$, *i.e.* $\frac{\text{man}}{\text{rational}} = zx + v\bar{z}\bar{x}$ as indicated above. That is, this indefinite value appears when we express 'man ÷ rational' in terms of 'man' and 'rational': but, as Mr Johnson points out, our result is like the traditional one, definite, when we express the same fraction in terms of 'animal'.

common terms, and too much restricted in their acquired signification, for it to be quite convenient thus to appropriate them. To make them fit our purpose we should not only have to confine them entirely to the extension of our classes irrespective of their meaning or intension, but we should have to insist upon assigning to Abstraction an indefiniteness of limitation to which it has never been accustomed in Logic[1].

Before concluding this chapter it will be well to call attention to a few considerations in the use of our symbols, which will be found to conduce to the convenience or accuracy of their use. Some of these are obvious enough when pointed out. For instance, in the employment of the additive sign (+) we laid it down that the common part of the connected terms was not to be counted twice. If however, with Boole, we were to adopt the non-exclusive notation this no longer holds good. Accordingly, in the original statement of our propositions, we should then take the precaution of so expressing our formulæ as to guard against this[2]. Instead of putting $x+y$, we write $x+\bar{x}y$. But in the course of working

[1] The only other distinct attempt at the systematic introduction of fractional forms into the treatment of Logic that I remember to have seen is by Bardili (*Grundriss der ersten Logik, &c.*, 1800). The signification he there gives to the signs of subtraction and division is however of far too metaphysical and non-logical a kind (in any sense of the word Logic with which we are now concerned) to deserve discussion here. Some account of his system will be found in Erdmann's *Geschichte der neuern Philosophie* (III. 479). It may be remarked that a succession of logicians (Jäger, Lichtenfels, Procházka, Kaulich:—see the historical notes at the end) have adopted the unfortunate device of indicating particular propositions by a fraction. Thus S standing for 'All S' they put $\frac{1}{S}$ for 'some S'; the idea being that we thus make it plain that we are dealing with a part only of S. But why S is placed in the *denominator* of the fraction passes mathematical comprehension.

[2] The first proposer of the 'x or y', as contrasted with the 'x or $\bar{x}y$' form, was probably Leibnitz; for

out results we shall often find that "double counting", in an obvious numerical form, is forced upon us. Are such results to be regarded as intelligible?

It is soon seen that there is no harm in passing through such inappropriate expressions, provided they do not occur in our final results. I may heap up one such term upon another, provided there is some expression at the end which shall neutralize the surplus. Thus $2xy$ is unauthorized by itself, and therefore $(x+y)-2xy$ cannot, as it stands, be read off into strict logical language. But a different arrangement makes it irreproachable, for it can be thrown at once into the form $x(1-y)+y(1-x)$, or $x\bar{y}+\bar{x}y$. It then becomes 'x or y, but not both'.

here, as on so many other logical points, we find that he had already proffered some suggestive hints. "$A+B \infty L$ significat A inesse ipsi L, vel contineri a L...Etsi A et B habeant aliquid commune, ita ut ambo simul sumta sint majora ipso L, nihilominus locum habebunt quæ hoc loco diximus aut dicemus" (*Specimen demonstrandi*, Erdmann, p. 94). He had clearly realized the fact that we cannot subtract in Logic except when we are dealing with mutual exclusives ("incommunicantia"). Thus, "Si $A+B \infty C+D$, et $A \infty C$, erit $B \infty D$, modo A et B itemque C et D sint incommunicantia" (Erdmann, p. 97. The sign ∞ corresponds to our $=$). Again on the same principle he solves the problem, "Sit $A \infty A$, dico reperiri posse duo B et N sic ut B non sit ∞N et tamen $A+B$ sit $\infty A+N$" (*ib.* 97).

Ploucquet again says much the same. Having obtained the result $nA+nA=m+b$, he goes on; "da aber $nA+nA$ nur eine Zeichenwiederholung, nicht aber eine Sachwiederholung ist [a clear and terse way of putting it] so kommt $nA=m+b$". (*Sammlung*, p. 254;—but the context involves several misconceptions.)

To those who looked upon A and B as standing for *notions* or *attributes*, rather than for classes, it seems to me that such a view as this was most natural, and indeed almost obligatory; the non-repetition of the attribute is only a case of "the one in the many". Lambert, however, so far as he is explicit here, rather takes the opposite view:—"man drücke die eigenen Merkmale des a durch $a \mid b$ aus, und die eigenen Merkmale des b durch $b \mid a$; so hat man $a \mid b+b \mid a+ab+ab=a+b$" (*Log. Abhandlungen*, I. 11).

Similar remarks apply to the use of the subtractive sign. We cannot intelligibly deduct except from a class which obviously contains that which is to be deducted. But there is no harm in beginning with one or more negative terms, provided we set matters right before the conclusion by the insertion of the requisite terms from which they could have been deducted. It should be observed that in doing this we are only carrying out more boldly and consistently what common language has already recognized as convenient. Thus we might say that 'except hotel keepers, muleteers and beggars, the Swiss are an agreeable people' though in perfect strictness we could not 'except' from what had not been already laid down. Symbolic Logic does nothing more than carry out fully and consistently this right of regarding the order of our terms as indifferent in this respect. We may group them as we please, provided the aggregate is capable of falling into an intelligible arrangement. Thus $xy + xz + yz - 3xyz$ is really only another way of 'saying' in symbols, $(1 - x)yz + (1 - y)xz + (1 - z)xy$; which is itself a way of saying, in words, 'whatever belongs to two and two only of the three classes x, y, and z'. This is therefore merely an extension of the right which even common language has found it expedient to claim in certain cases and to a partial extent.

The next simplification or generalization to which I will call attention is less obvious, at least in some of its applications. The reader will remark that throughout our explanation of the symbolic forms xy and $\frac{x}{y}$ we have never said anything to imply that x and y must be merely single logical terms. We have spoken of them as representing logical classes; and our explanation, being a purely logical one, will therefore cover the case of x and y

representing any kind of logical class. In the case of the 'multiplication' of terms, indeed, this is readily recognized, and we have had examples in point in the last chapter. All that remains therefore is to call attention to a few peculiar or limiting cases which result from such an admission.

When we combine classes which are composite in their character, that is, which are built up of a plurality of terms, we generally, whilst restricting the actual limits of the class, increase the number of terms by which it is expressed. Thus $(a+b)(c+d)$, or the class common both to 'a and b' and to 'c and d', is assigned by the four elements $ac+ad+bc+bd$. This class is presumably narrower than either $a+b$ or $c+d$, but it contains twice as many terms. Sometimes however such a process of combination will cause a number of elements to cancel each other and disappear, and so the resultant class may be simpler in symbolic expression as well as narrower in actual extent. Thus[1] combine $(a+\bar{a}\bar{c})$ with $(ac\bar{e}+\bar{a}ce)$ and we have merely $ac\bar{e}$; that is, in words, the class common to 'what is a, or neither a nor c' and to 'what is ac and not e, or ce and not a' is simply 'ac that is not e'.

It is quite possible that the two classes thus combined may contain no common part at all, in which case the process of multiplication results in zero. When classes are thus mutually exclusive, indeed, this multiplication may sometimes be the simplest way of proving the fact. Thus, for instance, $(a\bar{c}\bar{e}+\bar{a}c\bar{e})(ac\bar{e}+\bar{a}\bar{c}e)=0$, for there are no members common to both.

The same consideration applies also to our fractional expressions. Thus, in accordance with the general formula, in its simpler form,

$$\frac{x}{y}=x+v\bar{y}\,(\text{with } x\bar{y}=0\text{; as a condition of interpretation})$$

[1] The abbreviation $\bar{a}$ for $(1-a)$ will be ordinarily adopted in future.

we may deduce

$$\frac{a\bar{c}+\bar{a}c}{1-e}=a\bar{c}+\bar{a}c+ve: \text{ with } a\bar{c}e+\bar{a}ce=0.$$

The verbal description of this is that the most general expression of 'the class which, on restriction by taking only that part of it which is not e, shall just be reduced to a only or c only of the two a and c', is 'a or c only, together with "we know not what" of that which is e'.

Many other forms might be suggested, some of which will look very strange to those whose associations with fractional forms are confined to division and to representation of ratios. Thus the form $\frac{0}{1-xy}$ yields easily enough $0+v.xy$, or $v.xy$. The explanation of this is simple enough, the expression being merely a roundabout statement of the familiar Law of Contradiction. We are asked to assign the class such that its combination with $1-xy$, or not-xy, shall give *nothing*. The answer of course is that 'xy, or any part whatever of xy' will be thus exclusive of 'what is not xy', and this is therefore the most general assignment of the class in question.

The only caution to be kept in view here is that both the numerator and denominator of our fractional form shall be intelligible class expressions. We have no right, for instance, to put together such a form as $\frac{x-y}{x-z}$ unless we mean to imply that both y and z are parts of x. A formula will indeed be given in a future chapter which is competent (under certain conditions) to deal with such expressions as these, and to force an explanation out of them. But so long as we confine ourselves, as at present, to simple logical explanations, we must not try to extend these expressions beyond the limits

within which we can clearly accept and interpret them. So long as x and y are logical classes, but only so long, may we regard $\frac{x}{y}$ as equivalent to $xy + v.\bar{x}\bar{y}$, and as implying also the condition $x = xy$. If, within these limits, we find the formula lead to falsity or absurdity, it can only be that we were asking a question which was in itself in some way false or absurd.

The equivalence of the two forms,—the fraction, and its logical rendering,—it must be remembered, is complete: provided of course the condition of interpretability is introduced. That is, $\frac{x}{y}$ and $xy + v.\bar{x}\bar{y}$ involve precisely the same implications in respect of what they assert and deny, so that either may be substituted for the other. It is not a case of mere implication or derivation, but of equivalence or convertibility.

It may also be pointed out here that, in Logic, any such fraction and its reciprocal, when expressed in their fuller form, will be found to yield the same symbolic result. Thus $\frac{x}{y} = \frac{y}{x}$, for each may be converted into $xy + v.\bar{x}\bar{y}$. Of course their *value* is not the same,—that is, their logical import,— for the preliminary conditions of interpretability which they respectively involve, are different. In $\frac{x}{y}$ the condition is $x\bar{y} = 0$, and in $\frac{y}{x}$ it is $\bar{x}y = 0$: that is, the relations of x and y to each other are different in the two cases. When the results are expressed in their simpler forms, as $x + v.\bar{y}$, $y + v.\bar{x}$, the semblance of identity of course disappears.

In the case of this 'divisive' process we may repeat still more decidedly what was said in the last chapter, as to the

non-necessity, for the mere purposes of calculation or inference, of the subtractive process. And it must be admitted that we cannot here offer the plea that we are doing no more than putting into symbols what popular thought has already fully accepted. But none the less its omission is, in my judgment, a speculative loss. If logical processes were, like book-keeping say, simply means to a practical end, I quite admit that any exponent of the subject who should insist upon a full discussion of four rules, when his readers could get through all their work with two, would be laying a needless burden upon weary shoulders. It is indeed quite possible that those who can recall the mental effort, and the consequent sense of satisfaction experienced, in working out the real signification of formulæ which the inventor of them had regarded as strictly uninterpretable, may overrate the value of such discipline. But, making all due allowance for this bias, I do most strongly maintain the great speculative advantages of thoroughly working out all these processes, and of realizing their mutual relations. And I cannot but think therefore that those who discard them from their treatment lose more than they gain by the consequent simplification.

What then is the substitute for this particular inverse process? It may be expressed, in the simplest form, thus:— If xy is a, then x is not-y or a: or, in a slightly more complicated form, if xy is a or b, $x\bar{a}$ is $\bar{y}$ or b. That is, a term may be transferred from one side to the other, provided we not only change the sign (*i.e.* contradict the term), but also change the symbol of connection, by substituting + in the predicate for × in the subject, and conversely. In this latter form it is sometimes called Peirce's rule. It is adapted, we must remember, not to the equational rendering of propositions adopted here,—for which purpose Boole's rule, or our

simplified equivalent, is more suitable,—but rather to the implicational or predicational renderings. Under the first of these forms it is largely used by Schröder, Peirce, and others: under the latter by Dr Keynes.

It will be observed that the peculiar indefinite term $(v\bar{x}\bar{y})$ does not here make its appearance. But this is only because in implication, subsumption, or predication, the predicate is always to be understood in an indefinite application. When we obtain, as one side of an equational statement, $xy + v\bar{x}\bar{y}$, we thereby state that we include the *whole* of xy and 'some' of $\bar{x}\bar{y}$; but when we have the predicate form 'x is $\bar{y}$ or a', both $\bar{y}$ and a are put on the same footing of indefiniteness as $v\bar{x}\bar{y}$. This distinction will be more fully insisted upon hereafter.

It is a significant illustration of the narrow range and conventional limitation of the old logic that it does not seem ever to have raised this question, simple as it is on the equational and simpler still on the customary predicational rendering. Obvious as the question seems no one appears to have asked, 'If x that is y is z, what is x in general'? Without venturing to say of any particular doctrine that it is not to be found somewhere in the vast field in question, one may say with some degree of confidence that this is not to be found where one would expect to find it, viz. in the elementary hand-books.

The only remaining generalization which we shall find it convenient to introduce into Logic is the mathematical term '*function*'. There is nothing in this which should raise any objection. We are doing absolutely nothing more than making use of a somewhat wider generalization of the same kind as those with which the ordinary logician is already familiar, and which form one of the main distinctions between his language and that of common life. We are accustomed

to put an X or a Y to stand not for this or for that subject or predicate only, but for subjects and predicates generally; and so we put *Barbara* or *AAA* to stand for one particular form of argument whatever its matter may be. This does well enough for such simple terms and propositions as the common Logic mostly has to deal with; but when we come to grapple with more complicated terms and propositions we shall find a need for some corresponding advance in our technical language. We want some kind of expression which shall stand for any class-group or class-equation, however complicated these may be, provided only they involve some given term as one of their constituent elements.

For instance:—'Every XY which is either A or B', 'No AB which is not X, is Y':—here we have respectively a class-group and a class-equation, both of which involve the term X. They are of course already in what is called an abstract form, as compared with the concrete language of ordinary life; but inasmuch as they both involve the common element X, we may make a higher abstraction out of them in respect of this element. We do this in calling them both 'functions of X'. Writing them respectively; $xy(a + \bar{a}b)$, $ab\bar{x}y = 0$, we may use the common form $f(x)$ to stand for them both. We can, of course, mark the fact that the latter is an equation by writing it $f(x) = 0$, and this it will generally be convenient to do.

Technical language is, it must be remembered, called for by technical requirements. The reason why we want a common form for such various expressions is to be sought in the fact that we propose to subject them all alike to certain common operations. Ordinary Logic finds no occasion to do this, and therefore does not require to take note of that common element in them which gives ground to such operations, and which we indicate by the expression $f(x)$ or 'function of x'.

This necessity for the performance of common operations is of course the reason why we call such an expression a function of x rather than of anything else. It will commonly involve other terms besides x, and therefore be a function of them also. The sole reason why we single out one element and regard it as $f(x)$, is that x, and not one of the other terms, is the common element in virtue of which we propose to operate upon it and to which alone therefore we pay attention for the time being.

It may be added that, on the view adopted in this book, $f(x)$ never stands for any thing but a logical class. It may be a compound class aggregated of many simple classes; it may be the class indicated by certain inverse logical operations; it may be composed of two groups of classes declared equal to each other, or (what is the same thing) their difference declared equal to zero, that is, a logical equation. But however composed or derived, $f(x)$ with us will never be anything else than a general expression for such classes of things as might fairly find a place in ordinary Logic.

CHAPTER IV.

ON THE CHOICE OF SYMBOLIC LANGUAGE.

IN the last two chapters we have shewn how the familiar symbols of mathematics may be used to express logical relations and processes. The ease and accuracy with which they do this will to many minds afford a complete justification of their employment for this purpose, but as repeated and forcible protests[1] have been raised in various quarters against introducing them into Logic, a short chapter of justification may conveniently be inserted here.

That a language of symbols of some kind or other is needed must I think be taken for granted. Such a language indeed is already admitted into the ordinary logic just so far as that science is supposed to need it: and, on the same ground, if more of it is wanted more of it must be called for. If we are to handle such classes, for instance, as that composed of "A which may be both B and C, but not one only of the two except when it is either D or E or both," what arguments can be urged for trying to work with this cumbrous verbal

[1] *e.g.* Spalding's *Logic*, p. 50. T. S. Baynes's *New Analytic*, p. 150. Dr Keynes has shewn, by his *Formal Logic*, that the most complicated examples yet proposed can be solved with much less of symbolic apparatus than had hitherto been supposed possible.

expression except such as would tell equally against condensing some complicated statement into the form '*All X* is *Y*'? And, as the reader knows by now, what we have to be prepared to do is to manipulate groups of terms such as that just offered. Of course if we only proposed to go over the familiar ground again in new words, and were asking for a fresh array of symbols in which to exhibit the common Conversion, Opposition, and Syllogism, of the ordinary Logic, such a demand might reasonably be protested against. It is indeed true that most of the earlier symbolists did not propose to do more than this, but considered that the transformations they effected would receive their full justification when they could show how we might go in detail through all the moods of the syllogism and exhibit them clothed in symbolic language. Probably some logicians still believe that nothing more than this has even yet been attempted, but it would be waste of time to stop to argue against such a misapprehension as this, after what has been said in the preceding chapters.

The question therefore is considerably narrowed. Our choice lies between taking the language of mathematics as far as this will serve our purpose, or inventing a new one, for there is certainly no existent rival to the former. That is, there is no complete system of symbolic language, devised for any other science, which would practically answer our purpose. It must be admitted that the choice is not altogether without difficulty.

The prevalent objections to employing mathematical symbols rest partly upon some misapprehension of their nature and existent range of interpretation. Something has been already said on this subject in the Introductory Chapter, so that a few words will suffice here. The objectors who protest against the introduction "of relations of number and

quantity[1]" into logic, and who reject the employment of the sign (+) "unless there exists exact analogy between mathematical addition and logical alternation[2]", cannot, it would appear, get rid of the notion that mathematics in general are of the nature of elementary arithmetic. We must again remind the reader of the wide range of interpretation which already exists within the domain of mathematics: how the sign (+) starting as the sign of addition in ordinary arithmetic, has, in algebra, come to cover the case of subtraction; and has continued to extend its range of application till in Quaternions, A and B indicating both the magnitude and the direction of two steps, $A+B$ will indicate the net result produced by taking successively first one and then the other of these steps. Again the sign ($\times$) has similarly extended its interpretation; beginning with true multiplication of integers, it has embraced fractions and negative quantities within its rules, and has continued extending its signification till it too has become transformed in Quaternions; so that finally $A \times B$ may mean, not multiplication, but (amongst other things) a certain rotation of a line through an angle. Similarly the sign (=), when applied to vectors, denotes both equality of length, and parallelism of direction[3].

[1] Spalding's *Logic*, p. 50.

[2] Jevons's *Principles of Science*, p. 68.

[3] The mathematical reader is recommended to consult a paper by Mr Spottiswoode on "Some recent generalizations of Algebra" (*Proc. of London Math. Soc.* Vol. IV. p. 147). He there says:

"In the majority of systems proposed, the distributive and associative principles have been adopted [certain exceptions being mentioned even here] and starting from this basis a variety of laws of multiplication might be laid down. The following apparently comprise the principal systems now in use:—

(1) The Commutative principle might be adopted, so that $\iota_1, \iota_2, \ldots$ being the units, $\iota_1 \iota_2 = \iota_2 \iota_1$; and the actual value of such a product might be the subject of any other arbitrary assumptions. Such an algebra might be called commutative.

(2) The Commutative principle

It is clear that these considerations decidedly blunt the edge of much of the objection in question. It is no longer a case of shifting a term or sign from one precise and rigid signification to another, but that of extending in a new direction the signification of signs which have already proved themselves able and willing to undergo a succession of very important extensions.

Another objection, and in some respects a more reasonable one, against the practices of the symbolic logician, would be directed not at the *interpretation* he puts upon the symbols but at his meddling with their actual *laws of operation*. It may be urged, for instance, that we do not use these mathematical signs consistently, that is, that we put special restrictions upon their laws of operation which are not admitted in mathematics:—that, for instance, we maintain that x^2, as also every higher power of x, is the same as x itself, and that, as a consequence of this, we cannot adopt the plan of striking out common factors from the numerator and denominator of a fraction. This is so: we do depart in these respects from the ordinary practice of the mathematician in most of his departments. But here again a little reflection upon what is already admitted somewhere or other in mathematics will weaken such an objection. Do we depart further from the primary traditions of arithmetic than the Quaternionist does? It is a question if we can be said to depart so far, for at least we still adhere[1] to the 'commutative law' that $xy = yx$, whilst he finds it necessary to reject it, and assigns a different interpretation

being suspended, the following relation might be adopted: $\iota_1 \iota_2 = - \iota_2 \iota_1$, expressive of what might be called the Alternative principle". Two other possible cases are then mentioned.

[1] Wundt (*Logik*, I. 225) has been already mentioned as an exception here. See Ch. III. p. 86.

to these two expressions. With him they are, generally speaking, not equivalent.

Since, then, there is already so considerable a license in these respects admitted amongst mathematicians, there is not so much fear that the logician will unsettle the minds of men, or introduce misleading associations, if he decides to employ his symbolic language in the way which he thinks will suit him best. The question is one of a balance of opposite advantages, there being something to be said for and against each side. It must be remembered that there are but very few signs which we find it convenient to borrow: in fact only the following:—four symbols for operations, $+$, $-$, $\times$, $\div$; the sign of equality $(=)$ and two symbols for quantities, 1 and 0:—these being for the most part just those oldest and most general symbols which have already undergone the widest transfer or generalization of interpretation. The abbreviated expression for *function*, viz. (f), need hardly be formally included in the list, since we can scarcely be considered to change or extend its signification.

The reason which makes me decide in favour of the plan of employing the symbols of mathematics is briefly this. The introduction of any new set of symbols is in itself somewhat of an evil. Symbolic language ought if possible to be used with mechanical facility, and this presupposes a considerable amount of practice. Every one who has learnt a system of shorthand knows what a length of time elapsed before it ceased actually to frustrate its main object, by causing rather than avoiding trouble. Now the *interpretation* of our symbols is only occasional whilst their *employment* is by comparison constant. Interpretation is demanded at the primary step of writing down our data, and at the final step of reading off the answer, but along the path between the start and the finish we are not generally obliged

to think of interpreting our formulæ at all. Accordingly when the alternative is before us of inventing new symbols, or only assigning some new meaning to old and familiar ones, experience and reason seem rather in favour of the latter plan. Certainly the experience of the mathematicians appears to tell in this direction, which ought to count for much. When they have to denote a new conception or a new law of operation, of course they may want a new symbol for it. But when the law of operation is the same, or even partially the same, they continue to use the old symbol even though the signification may have undergone a very considerable change. To take then one of the simplest instances: which is easier, to use the familiar sign (+) as we have always been accustomed to use it, bearing in mind, as we do so, that $x + y$ does not mean *addition* of x to y (an early prejudice which the mathematician has long laid aside), or to invent a new symbol (say ·|·) where we have to learn anew both the laws of operation and the signification?

Whilst then we shift the signification of the symbols we retain their laws of operation as far as possible unchanged. Indeed the only change we venture to make is of the nature of special limitation rather than of actual alteration. Thus to identify x^2 with x is necessary in certain cases even in mathematics, for instance when $x = 1$; and to forbid division by x is also necessary, in case $x = 0$. We can hardly be said therefore to transgress any universally binding usage. As an illustration of an unjustifiable use of such symbols a practice (which has however the sanction of several good names) must be noticed, that, namely, of employing + and − to mark respectively affirmation and denial. The analogy on which this usage is founded is very slight, amounting indeed to little more than the fact that two denials (in the case of strict contradictions) result in re-

affirmation, just as $(-)$ twice repeated yields $(+)$. But the commutative law is here rejected, for if $X + Y$ means that 'All X is Y' we must clearly refuse to identify this with $Y + X$. Again, if we express 'if S, then not P' by $(+S - P)$, we might be tempted, following familiar usage, to regard this as equivalent to $(-P + S)$ which would be, of course, to fall into a familiar fallacy[1].

Those who propose a new notation commonly, and not unnaturally, assume that it will supersede all others. But those who approach it as strangers know that the odds are decidedly that it will only prove one more of those many innovations which perplex and weary the lecturer, historian, and critic. Hence we are tempted to use the argument, dear to those in authority, that if we loosen the sanctions of orthodoxy heresies will multiply. Only those whose professional employment compels them to study a number of

[1] As indicated above, the actual usage here is various. Maimon made + and − equivalent respectively to 'is' and 'is not', so that '$a+b$' meant 'a is b', and '$a-b$' meant 'a is not b'. As Darjes used them they might be best rendered by 'posit' and 'sublate', for he affixed them to each term. Thus $+S-P$ meant 'posit S and we sublate P'. Drobisch's use is different, as he seems to confine them to mark propositions as wholes: thus $+u, -u, +p, -p$, stand respectively for what are commonly indicated by A, E, I, O. But in all cases alike the usage seems to me faulty and misleading.

For downright grotesque perversion of mathematical terms some of the non-mathematical logicians are unequalled. Many readers must have been puzzled by Hamilton's symbolic equivalent for the Law of Contradiction. "This law is logically expressed in the formula,—what is contradictory is unthinkable. $A =$ not-$A = 0$, or $A - A = 0$" [*Logik*, I. 81:—Is not this, by the way, an attempt at rendering a passage in Bachmann (*Logik*, p. 43), "Reine position und negation, setzen und aufheben, $(+A - A)$, in einem Denkakte unmittelbar verbunden, vernichten sich, weil sie einander rein entgegengesetzt sind $(A - A = 0)$"?] Mr Chase again (*First Logic Book*) making + and − do duty for affirmation and negation employs the negative particle *as well*, writing e.g. *Cesare*, thus:

No $Z - Y$,
All $X + Y$,
No $X - Z$.

different works have any idea of the bewildering variety of notation which is already before the world. A new notation is not like a new fact or theory from which, so to say, the passer-by may pick up something to remember. It is meant for habitual use, and thus practically aims at the exclusion of all rivals[1].

There are two subordinate advantages in employing an already widely-used and familiar set of symbols. One of these is in their occasional *suggestiveness.* Take for instance that inverse process to class-restriction which was explained in the last chapter. It is not by any means an obvious process, and though perfectly intelligible in itself we can hardly believe that it would have suggested itself to the mind except by way of the symbols. We are very familiar with the particular inverse process of Division in relation to Multiplication, and when we use the latter sign to denote class-restriction the enquiry seems forced upon us to determine what there is in Logic corresponding to the former. As soon as we write down $xy = z$, we can hardly refrain from

[1] In order to gain some idea of what has been from time to time proposed in this way, the reader may turn to the final chapter of this volume, where he will find a detailed account of over thirty distinct notations for that simplest of propositions:—the Universal Negative.

The following illustrations may be given here. The same *meaning,*—the distinction between a term and its contradictory;—has been variously symbolized as follows:

A a (De Morgan, Jevons),
a $\bar{a}$ (Boole, R. Grassmann),
a a' (Delbœuf, Mc Coll),
a $1-a$ (Boole),
a a_1 (Schröder),
a $-a$ (Segner),
a na (Maass),

whilst the same *symbols*:—capital and small letters respectively:—have been made to do duty for the following meanings:

The class A and its contradictory (De Morgan, Jevons),

The class A distributed and undistributed (Ploucquet),

The concept and its extension (Maass, *Logik*, p. 100),

The determined and the determining concepts (Wundt, *Logik*, p. 223),

Universal and particular propositions (Gergonne).

going on to try $x=\frac{z}{y}$, and then the question of the interpretability of the latter is forced upon us. Every mathematician knows what a fertile source of new theorems is found in the attempt to ascertain the analogues to such and such a familiar process in some other branch of analysis. Of course we must not permit such hints as these to be anything more than hints, for every logical rule must be established on its own proper grounds; but even hints may be of great value.

It may be remarked that the analogy of division just mentioned was perceived from the first by logical symbolists; (as was shewn, in the preceding chapter, in the case of Lambert and Holland). As they interpreted the step indeed, viz. as denoting Abstraction, the logical process was one which was already quite familiar, so that very likely the symbolic step was first suggested and justified by the logical. As *we* feel bound to interpret it, however, viz. in respect of extension or denotation, the case is very different. As just remarked, the step is not an easy one to grasp, and it is very doubtful if we should have been able to see our way to it without the help of the slight pressure in the right direction afforded by our wish to justify and explain an already familiar symbolic procedure.

Again, it seems really important to impress upon the mind of the student certain characteristics of symbolic language. The distinction between the mere laws of operation and the interpretation to be assigned to them, is apt to be overlooked, and this will very likely be still more the case if we insist upon introducing a new notation for one special class of interpretations. We may thus lose our appreciation both of the generalized extent over which the same laws of operation can prevail, and of the very various

though connected significations which this extent of application will serve to cover.

Various other ways have been adopted in order to prevent any confusion between the special logical usage of symbols and the ordinary mathematical usage of the same, and yet not to lose sight of the common properties. Thus Mr C. S. Peirce at one time proposed[1] to differentiate the logical use by the insertion of a distinguishing mark (a comma underneath). Instead of writing $a \cdot\!|\!\cdot a = a$, with Jevons, in order to represent the fact that logical 'addition' does not double the number of the common members, he wrote it $a \underset{,}{+} a \underset{,}{=} a$. This would be interpreted to mean that 'the (logical) addition, or aggregation, of a to a is (logically) equivalent to the class taken simply.' So again, as a consequence, if we know that a and b are mutually exclusive we have the formula $a \underset{,}{+} b \underset{,}{=} a + b$; for in this case the results of the logical and the arithmetical additions correspond.

Far the most important departure from the notation we have adopted is however that which is deliberately employed to indicate rejection of the equational rendering of our propositions. This, of course, principally concerns the copula (=). Those whose system is mainly founded on the notion of implication (i.e. of one proposition by another, as with Mr McColl) or on that of subsumption (i.e. of one class under another, as with Mr Peirce and Professor Schröder) not inconsistently reject the symbol of equality[2]. They either, with the former, invent an arbitrary symbol:—thus Mr McColl represents 'All a is b' by the sign $a : b$; though here he intends to remind us of a mathematical analogy, since this is the symbol of a *ratio*. Or, with the other

[1] (*Proc. of American Acad. of Arts and Sc.* 1867, p. 250.)

[2] Except, of course, for reciprocal implication or subsumption, of A by B and of B by A, when they write $A = B$.

writers, they introduce a symbol which by its construction shall intimate analogies. Thus Mr Peirce employs the symbol $-\!\!<$ and Professor Schröder employs $\subseteqq$. The intimation here suggested will be understood at once by reference to the first chapter, in which we discussed the schedules of propositions. It was shewn that the ordinary 'All A is B' covered the two actual class relations of identity of A with B, and of inclusion of A by B. The two symbols just mentioned are intended to remind us of this, as being compounded of, or modifications of, signs of identity and inclusion. Professor Schröder purposely employs $\subset$ rather than $<$, to shew that we are not concerned, as in mathematics, with the quantitatively greater and less.

Or again, an entirely new set of symbols might be introduced, intended to be applied not to logic separately in contrast with mathematics, but made so general as designedly to cover both. Among those who have actually offered something in this way, of a logical kind, may be noticed H. Grassmann. He proposes the use of a symbol $\frown$ to denote 'connection' (Verknüpfung) in general, and $\smile$ to denote its inverse, so that $a \smile b$ denotes the form which when joined by $\frown$ to b will yield a. He does not indeed in his definition refer to anything outside the domain of mathematics, but his language seems intended to be perfectly general: "By a general science of symbols [Formenlehre] we understand that body of truths which apply alike to every branch of mathematics, and which presuppose only the universal concepts of similarity and difference, connection and disjunction" (*Ausdehnungslehre*, p. 2). As a result of this generalized use we shall have to notice, in another chapter, a curious anticipation, in certain respects, of one detail in Boole's procedure.

It is to this generalized symbolic language, much as we

are here employing it, that some writers have applied, by a revival of an old word, the term *Algorithm*. Thus, for example, Delbœuf entitles his work, written on the same kind of subject as the present, "Logique algorithmique." There is no objection whatever to the word, but I have preferred to speak of "Symbolic Logic" as being more familiar in our language: 'symbolic,' as I understand it, being almost exactly the equivalent of 'algorithmic.'

There is also another old term which will be familiar to readers of Leibnitz and Wolf,—Characteristic,—which seems to me to cover much the same ground as Algorithmic and Symbolic: the word is thus defined by Wolf (*Psychologia empirica*, § 294 *seq.*) "Ars characteristica appellatur ea quæ explicat signorum, in rebus aut earundem perceptionibus denotandis, usum. Ars hæc adhuc in desideratis est."......"In Algebra istiusmodi signa habemus pro quantitatibus; sed desiderantur talia in philosophia pro rerum qualitatibus."... "Ars illa quæ docet signa ad inveniendum utilia et modum eadem combinandi eorumdemque combinationem certa lege variandi, dicitur Ars characteristica combinatoria. Vocatur a Leibnitzio etiam speciosa generalis."......"Si quis mente perpendit qualis numerorum in Arithmetica, magnitudinum in Algebra, syllogismorum in Logica, notio præsupposita fuerit antequam characteristica ad numeros magnitudines et syllogismos applicari potuerit, et quamdiu in Arithmetica atque Algebra commodi desiderati fuerunt characteres; is difficultatem artis characteristicæ combinatoriæ generalis haud difficulter æstimabit."

These extracts seem to indicate a tolerably clear appreciation of the end to be aimed at in constructing a generalized symbolic logic, but most of the discussions on this subject are much mixed up with the wider question of a general Philosophical Language. As the reader very likely knows,

this problem was keenly discussed in the seventeenth and eighteenth centuries and occupied the attention of Leibnitz more or less throughout his life. Speaking from a moderate acquaintance[1], I should say that what was mostly contemplated by the writers in question was more what we should now call either a universal language, or a general system of shorthand, than a logic. I mean that they do not, generally speaking, attempt any analysis of the reasoning processes; and that the words or symbols proposed by them do not stand perfectly generally for any classes whatever, like our x and y, but specially for such and such well-known classes as are already designated by general names; they differ, in fact, as Language in general does and should differ from Logic.

[1] I refer here to such works as the *Ars magna sciendi* of Athan. Kircher (1631); Bp. Wilkins's often mentioned *Essay towards a real character and a philosophical language* (1668) and Dalgarno's *Ars signorum* (1661). A discussion of Leibnitz's speculations and attempts in this direction will be found in Trendelenburg's *Historische Beiträge*, III. 1—48. Lambert, who took much interest in this subject, has discussed many of the special sets of symbols appropriate to particular arts and sciences. Of course the growth of international telegraphy, and other causes, have greatly varied the relative importance of the schemes known to him. (*Neues Organon*, *Semiotik*, § 1. Von der symbolischen Erkenntniss überhaupt.)

I have given a tolerably full discussion of some of these attempts at a Philosophical Language in chapter XXII. of my Empirical Logic. They are far more numerous than most persons suppose.

CHAPTER V.

DIAGRAMMATIC REPRESENTATION.

THE majority of modern logical treatises make at any rate occasional appeal to diagrammatic aid, in order to give sensible illustration of the relations of terms and propositions to each other. With one such scheme, namely that which is commonly known as the Eulerian, every logical reader will have made some acquaintance, since a decided majority of the modern familiar treatises make more or less frequent use of it[1]. Such a prevalent use as this clearly makes it desirable to understand what exactly this particular scheme undertakes to do, and whether or not it performs its work satisfactorily.

As regards the inapplicability of this scheme for the purposes of a really general Logic something was said by implication in the first chapter, for it was there pointed out how very special and remote from common usage is the scheme of propositions for which alone it is an adequate

[1] Until I came to look somewhat closely into the matter I had not realized how prevalent such an appeal as this had become. Thus of the first sixty logical treatises, published during the last century or so, which were consulted for this purpose:—somewhat at random, as they happened to be most accessible:—it appeared that thirty-four appealed to the aid of diagrams, nearly all of these making use of the Eulerian Scheme.

representation. To my thinking it fits in but badly even with the four propositions of the common Logic to which it is usually applied, but to see how utterly ineffective it is to meet the requirements of a generalized or symbolic Logic it will be well to spend a few minutes in calling the reader's attention to what these requirements are.

At the basis of our Symbolic Logic, however represented, whether by words by letters or by diagrams, we shall always find the same state of things. What we ultimately have to do is to break up the entire field before us into a definite number of classes or compartments which are mutually exclusive and collectively exhaustive. The nature of this process of subdivision will have to be more fully explained in a future chapter, so that it will suffice to remark here that nothing more is demanded than a generalization of a very familiar logical process, viz. that of dichotomy. But its results are simple and intelligible enough. With two classes, X and Y, we have four subdivisions; the X that is Y, the X that is not Y, the Y that is not X, and that which is neither X nor Y. And so with any larger number of classes. How then are these ultimate class divisions to be described?

For one thing, we can of course always represent the products of such a subdivision in the language of common Logic, or even in that of common life, if we choose to do so. They do not readily adapt themselves for this purpose, but when pressed will consent, though failing sadly in the desired symmetry and compactness. The relative cumbrousness of such a mode of expression is obviously the real measure of our need for a reformed or symbolic language. We must not however forget that we are not dealing with mathematical conceptions which common language will hardly avail to describe, but only with logical classes which can be completely and unambiguously designated by the traditional

modes of speech. However complicated the description of any given class may be, we could always build it up by means of X and not-X, Y and not-Y, and so forth; whether X and Y remain as letters, or be replaced by concrete terms.

But it need not be pointed out that we require something far more manageable and concise than this, if we wish to deal effectively with really complicated groups of propositions. For this purpose nothing better can be employed than letters such as we use in algebra. This is done of course to some extent in ordinary Logic, the only innovation upon which we have to insist being that of introducing equally concise symbols for negative terms. We could never work with not-x, in that form, and must therefore look about for some substitute. The full significant substitute is, as already shewn, $1-x$, and this will sometimes have to be employed. But it is too cumbrous for purposes of actual calculation. Of the various substitutes[1] that have been proposed for not-x we shall make a practice of employing $\bar{x}$.

The reader will see at once how conveniently and briefly we can thus indicate any desired combination of class terms, and, by consequence, any desired proposition. Thus xyz represents what is x, y, and z; $x\bar{y}\bar{z}$ what is x, but neither y nor z; $\bar{x}w(y\bar{z}+\bar{y}z)$ stands for 'what is not x, but is w; and is also either y but not z, or z but not y', and so forth. The significance of such expressions, when built up into propositions, will be fully discussed in a future chapter.

That such a scheme is complete there can be no doubt. But unfortunately, owing to this very completeness, it is apt

[1] The great objection to Jevons's plan, of using capital and small letters, is that it can only be applied to single terms, and not, as is often requisite, to complex groups of terms. The bar over an expression will not only conveniently meet these cases, but can be repeated twice or oftener by superposition :—the lecturer will quickly appreciate another disadvantage in the Jevons language when he has to speak it for long.

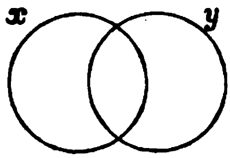
to prove terribly lengthy. The powers of 2 soon mount up; so that a pair of terms will yield 2^2 combinations, three will yield 2^3, or 8, and so on; the total number doubling every time. Of course in any particular proposition or problem we shall most likely not require to make appeal to more than a portion of these constituents, perhaps only to a few of them. But the existence of all has to be recognized, and a notation provided for every one of them. This is not a defect of symbolic notation in particular, but lies in the nature of things and the necessities of their mutual relations.

This then is the state of things which a reformed scheme of diagrammatic notation has to meet. Such a scheme must correspond in all essential respects to that regular system of class subdivision which has just been referred to under its verbal, and its literal or symbolic aspects. Theoretically, as we shall see, the desired aim is perfectly attainable. Indeed up to four or five terms inclusive, our plan works very successfully in practice; where it begins to fail is in the accidental circumstance that its further development soon becomes intricate and awkward, though never ceasing to be feasible.

I. On the proposed scheme we have to make a broad distinction, not recognized on the common scheme, between the representation of terms and the representation of propositions. We begin with the former. What we propose to do is to form a framework of geometrical figures which shall correspond to the table of combinations of x, y, z, &c. All that is necessary for this purpose is to describe a series of closed figures, of any kind, so that each in succession shall intersect all the compartments already produced, and thus double their number. That this successive duplication is what is done with the letter symbols is readily seen. Thus with two terms, x and y, we have four combinations; xy, $x\bar{y}$,

$\bar{x}y$, $\bar{x}\bar{y}$. Introduce the term z, and we at once split up each of these four into its z and its not-z parts, and so double their number. Provided our diagrams are so contrived as to indicate this, they will precisely correspond, in every relevant respect, to the table of combinations of letters.

The leading conception of such a scheme is simple enough, but some consideration is demanded in order to decide upon the most effective and symmetrical plan of carrying it out in detail[1]. Up to three terms inclusive, indeed, there is but little opening for any variety; but as the departure from the familiar Eulerian conception has to be made from the very first, it will be well to examine the simplest cases with some care. Our primary diagram for two terms is thus sketched :—

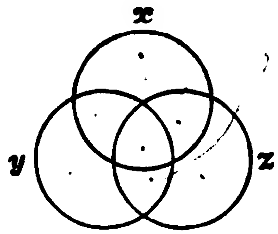


On the common plan this would represent a *proposition*, and is indeed commonly regarded as standing for the proposition 'some x is y'; though (as was mentioned in the first chapter) it equally involves in addition the two independent propositions 'some x is not y', and 'some y is not x', if we want to express all that it undertakes to tell us. With us however it does not as yet represent a proposition at all, but

[1] A brief historic sketch is given in the concluding chapter of some previous attempts, before and after Euler, to carry out the geometric illustration of propositions. I tried at first, as others have done, to illustrate the generalized processes of the Symbolic Logic by aid of the familiar method, but soon found that this was quite unsuitable for the purpose. Though the method here described may be said to be founded on Boole's system of Logic, I may remark that it is not in any way directly derived from him. He does not make employment of diagrams himself, nor does he give any suggestions for their introduction.

only the framework into which propositions may be fitted; that is, it indicates only the four combinations represented by the letter compounds, xy, $x\bar{y}$, $\bar{x}y$, $\bar{x}\bar{y}$.

Now suppose that we have to reckon with the presence, and consequently with the absence, of a third term z. We just draw a third circle intersecting the above two, thus:—



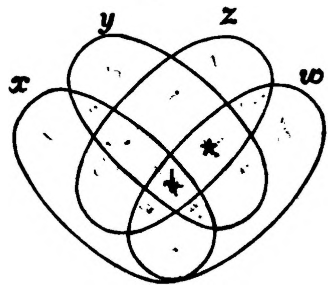
Each circle is thus cut up into four parts, and each common part of two circles into two parts, so that, including what lies outside of all the three, there are eight compartments. These of course correspond precisely to the eight combinations given by the three literal symbols; viz. xyz, $xy\bar{z}$, $x\bar{y}z$, $x\bar{y}\bar{z}$, $\bar{x}yz$, $\bar{x}y\bar{z}$, $\bar{x}\bar{y}z$, $\bar{x}\bar{y}\bar{z}$. Put a finger upon any compartment, and we have a symbolic name ready provided for it; mention the name, and there can be no doubt as to the compartment thereby referred to.

Both schemes, that of letters and that of areas, agree in their elements being mutually exclusive and collectively exhaustive. No one of the ultimate elements trespasses upon the ground of any other; and, amongst them, they account for all possibilities. Either scheme therefore might be taken as a fair representative of the other.

This process is capable of theoretic extension to any number of terms. The only drawback to its indefinite extension is that with more than three terms we do not find it possible to use such simple figures as circles; for four circles cannot be so drawn as to intersect one another in the way required. With employment of more intricate figures

we might go on for ever. All that is requisite is to draw some continuous figure which shall intersect once, and once only, every existing subdivision. The new outline must be so drawn as to cut every one of the previous compartments in two, and so double their number. There is clearly no reason against continuing this process indefinitely.

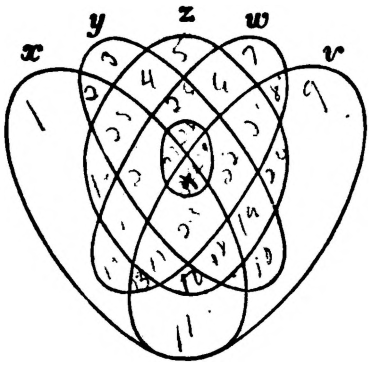
With four terms in demand the most simple and symmetrical diagram seems to me that produced by making four ellipses intersect one another in the desired manner:—



It is obvious that each component class-figure (say y) is thus divided into eight distinct compartments, producing in all 16 partitions; that these partitions are all different from each other in their composition, and are therefore mutually exclusive; and moreover that they leave nothing unaccounted for, and are therefore collectively exhaustive. And this is all that is required to make them a fitting counterpart of the 16 combinations yielded by x, y, z, w, and their negations, in the ordinary tabular statement.

With five terms combined together ellipses fail us, at least in the above simple form. It would not be difficult to sketch out figures of a horse-shoe shape which should answer the purpose, but then any outline which is not very simple and easy to follow fails altogether in its main requirement of being an aid to the eye. What is required is that we should be able to identify any assigned compartment in a moment. Thus it is instantly seen that the compartment marked with

an asterisk above is that called $\bar{x}yzw$. The simplest diagram I can suggest for five terms is one like this, (the small ellipse in the centre is to be regarded as a portion of the *outside* of z; i.e. its four component portions are inside y and w but are no part of z).



It must be admitted that such a diagram is not quite so simple to draw as one might wish it to be; but then it must be remembered what the alternative is if one undertakes to deal with five terms and all their combinations;—nothing short of the disagreeable task of writing out, or in some way putting before us, all the 32 combinations involved. I can only say for myself that having worked hundreds of examples, I generally resort to diagrams of this description, in order to save time, to avoid unpleasant drudgery, and to make sure against mistake and oversight. The way in which this last advantage is secured will be better seen presently, when we see how these diagrams are to be used in the representation of propositions as distinguished from that of mere terms or classes.

Beyond five terms it hardly seems as if diagrams, of the particular kind here described, offer much help, but then we have seldom occasion to trouble ourselves with problems which would introduce more than that number[1]. Although

[1] If we wanted to use a diagram for *six* terms (x, y, z, w, v, u,) one plan would be to take *two* five-term figures, one for the u part and the

however diagrammatic illustration fails to keep pace with true symbols, or letters, in respect of symmetry and simplicity, we must remember that they are really in strict correspondence with each other. No combination of class terms could be expressed either in symbols, or in common language, which could not be made sensible to the eye by a suitable diagrammatic construction.

We have endeavoured above to employ only symmetrical figures, such as should not only be an aid to reasoning, through the sense of sight, but should also be to some extent elegant in themselves. But for purely theoretic purposes the rule of formation would be very simple. We should merely have to begin by drawing any closed figure, and then proceed to draw others in succession subject to the one condition that each is to intersect once, and once only, all the existing subdivisions produced by those which had gone before. There is no need here to exhibit such figures, as they would probably be distasteful to any but the mathematician, and he would see his way to drawing them readily enough for himself[1].

other for the not-*u* part of all the other combinations. This would give the desired 64 subdivisions. Of course this loses the advantage, to some extent, of the *coup d'œil* afforded by a single figure. A far better method however is that described at the end of this chapter.

[1] It will be found that when we adhere to continuous figures, instead of the discontinuous five-term figure given above, there is a tendency for the resultant outlines thus successively drawn to assume, after the first four or five, a comb-like shape. If we begin by circles or other rounded figures the teeth are curved, if by parallelograms then they are straight. Thus the fifth-term figure will have two teeth, the sixth four, and so on, till the $(4+x)^{th}$ has 2^x teeth. There is no trouble in drawing such a diagram for any number of terms which our paper will find room for. But, as has already been repeatedly remarked, the visual aid for which mainly such diagrams exist is soon lost on such a path. It must be remembered however that their theoretic perfection, as regards the exclusiveness and exhaustiveness of the component portions, is unaffected by their intricacy.

A number of deductions will occur to the logical reader which it may be left to him to work out in detail. Some of them may be briefly indicated. For instance, any two compartments between which we can communicate by crossing only one line, can differ by the affirmation and denial of one general term only, e.g. $xyzw$ and $xy\bar{z}w$. Accordingly, when the two terms corresponding to such compartments come to be united, or as we may say, 'added', together, the result may be simplified by the omission of this term z; for the two together make up all xyw. Any compartments between which we can only communicate by crossing two boundaries, e.g. $xy\bar{z}w$ and $x\bar{y}zw$, must differ in two respects: it would need *four* such compartments to admit of simplification, the simplification then resulting in the opportunity of dropping the reference to *two* terms. For instance, $xy\bar{z}w$, $x\bar{y}zw$, $xyzw$, $x\bar{y}\bar{z}w$, taken together, amount simply to xw. In talking thus of crossing boundaries it must be remembered that to cross the same one twice is equivalent to not doing so at all, and that to do so three times is the same as doing so only once; it merely puts us outside if we were inside before. These considerations will be found to be of considerable significance when we come to the question of elimination.

II. So far then this diagrammatic scheme has only been shown to represent classes or terms, we have now to see how it can be worked so as to represent propositions. The best way of introducing this question will be to enquire a little more strictly whether it is really *classes* that we thus represent, or merely *compartments* into which classes may be put? The question is by no means an idle one, though its full significance will not be apparent until we have discussed the nature of Hypotheticals and the import of propositions generally.

The most accurate answer is that our diagrammatic sub-

divisions, or for that matter our symbols generally, stand for compartments and not for classes. We may doubtless regard them as representing the latter, but if we do so we should never fail to keep in mind the proviso, "if there be such things in existence." And when this condition is insisted upon, it seems as if we expressed our meaning best by saying that what our symbols stand for are compartments which may or may not happen to be occupied.

The reason for insisting upon this distinction is to be found in the absolute impossibility of ascertaining, until we have fully analysed our premises, whether or not any particular combination of terms is possible under the assigned conditions. We must be prepared to make provision for any number of terms, and it is impossible to foresee whether the existence of such and such a class is compatible with the data. We have a notation for each class; its place is ready for it; but will it be found there, or will the place be empty? Common Logic, dealing as it does with seldom more than two or three terms at a time, can evade the consequent difficulty, or can make tacit suppositions which will help to solve it in most cases. But with us the possible contingencies are far too numerous to be foreseen and provided against.

Take, for instance, the following group of premises, which are by no means of a very complicated nature:—

All x is either both y and z, or not y,
All xy that is z is also w,
No wx is yz.

It would not be easy to detect, from mere inspection of these data, that though, when taken together, they admit the possible existence of such classes as xz and yz, they deny that of the class xy. But since, as they stand, xy is the subject of one of the premises, we could not consistently admit such a conclusion as this unless we restrict the force

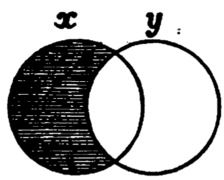
of that premise to what it *denies;* i.e., unless we confine ourselves to saying that it just destroys the class $xyz\bar{w}$, or 'x that is y and z but not w', and does nothing else. We find in fact that to consider ourselves bound to maintain the existence of all the subjects and predicates, instead of merely denying the existence of the various combinations destroyed, would sadly hamper us in the manipulation of complicated groups of propositions[1].

It would of course be pedantically at variance with ordinary usage to insist upon never speaking of anything but compartments. I shall therefore freely use such expressions as 'the class $xyzw$', and so forth. But when we thus speak in the course of our analysis it must always be understood that we do so without prejudice for or against the existence of such a class. The compartment necessarily exists, because it is purely formal, but it must be left to the data to decide whether or not it is occupied. However we like to phrase it, this distinction between an ideally perfect scheme of notation or classification which will meet every requirement, and its limitation by one possible class after another being proved incompatible with our data, must always exist. A complete enumeration of compartments is one thing, but it is quite another to be able to prove that there is a class of things available to put into each of them. If this were not so there would be no significance in our propositions; one way of measuring the significance of which is by comparing the number of potential and actual classes in our complete scheme.

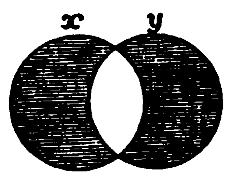
Our diagrams very readily lend themselves to mark this

[1] The grounds for thus interpreting the import of a proposition will be fully discussed in the course of the next chapter, and rules will be offered, dependent upon this analysis, for reading off any given proposition into its constituent denials.

distinction, and the plan we adopt for doing so is nothing but the representation of propositions. The full justification of the particular method here adopted must be reserved for the next chapter, but the present will be the best connection for giving a general description and illustration of it. What we do then, is to ascertain what combinations or classes are negatived by any given proposition, and proceed to put some kind of mark against these in the diagram. For this purpose the most effective means is just to shade them out. For instance the proposition 'all x is y' is interpreted to mean that there is no such class of things in existence as 'x that is not-y', or $x\bar{y}$. All that we have to do is to scratch out that subdivision in the two-circle figure[1], thus:—



If we want to represent 'all x is all y', we take this as adding on another denial, viz., that of $\bar{x}y$, and proceed to scratch out that division also; thus



On the common Eulerian plan we should have to begin with a new figure in each of the two cases respectively, viz., 'all x is y' and 'all y is x'; whereas here we start with the

[1] Other logicians (*e.g.* Schröder, *Operationskreis*, p. 10; Macfarlane, *Algebra of Logic*, p. 63) have made use of shaded diagrams to direct attention to the compartments under consideration; not, as here, with the view of expressing propositions.

same general outline in each case, merely modifying it in accordance with the varying information given to us.

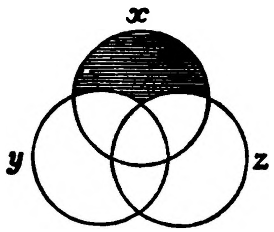
We postulate at present that every universal proposition may be sufficiently represented by one or more denials, and shall hope to justify this view in its due place. But it will hardly be disputed that every such proposition does in fact negative one or more combinations, and this affords an excellent means of combining two or more propositions together so as to picture their collective import. The first proposition empties out a certain number of compartments. In so far as the next may have covered the same ground it finds its work already done for it, but in so far as it has fresh information to give it displays this by clearing out compartments which the first had left untouched. All that is necessary therefore for a complete diagrammatic illustration is to begin by drawing our figure, as already explained, and then to shade out, or in some way distinguish, the classes which are successively abolished by the various premises. This will set before the eye, at a glance, the whole import of the propositions collectively.

How widely different this plan is from that of the old-fashioned Eulerian diagrams will be readily seen[1]. One great advantage consists in the ready way in which it lends itself to the representation of successive increments of knowledge as one proposition after another is taken into account, instead of demanding that we should endeavour to represent the net result of them all at a stroke. Our first data abolish, say, such and such classes. This is final, for, as already intimated, all the resultant denials must be regarded as absolute and

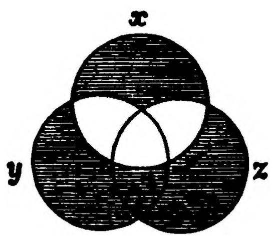
[1] I have not found any previous attempt to represent propositions on this scheme. Scheffler's elaborate *Naturgesetze* (Part III., on Logic) was apparently published about the same time as my paper in the *Phil. Magazine*.

unconditional. This leaves the field open to any similar accession of knowledge from the next data, and so more classes are swept away. Thus we go on till all the data have had their fire, and the muster-roll at the end will show what classes are, or may be, left surviving. If therefore we simply shade out the compartments in our figure which have thus been successively declared empty, nothing is easier than to continue doing this till all the information furnished by the data is exhausted.

As another very simple illustration of the contrast between the two methods, consider the case of the disjunction, 'All x is either y or z'. It is very seldom even attempted to represent such propositions diagrammatically, (and then, so far as I have seen, only if the alternatives are mutually exclusive), but they are readily enough exhibited when we regard the one in question as merely extinguishing any x that is neither y nor z, thus:—



If to this were added the statement that 'none but the x's are either y or z' we should meet this fresh assertion by the further abolition of $\bar{x}y$ and $\bar{x}z$, and thus obtain:—



And if, again, we erase the central, or xyz compartment,

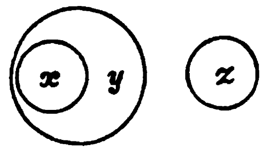
we have then made our alternatives exclusive; i.e., the x's, and they alone, are either y or z only.

Now if we tried to do this by aid of Eulerian circles we should find at once that we could not do it in the only way in which intricate matters can generally be settled, viz., by breaking them up into details, and taking these step by step, making sure of each as we proceed. The Eulerian figures have to be drawn so as to indicate at once the final outcome of the knowledge furnished. This offers no difficulty in such exceedingly simple cases as those furnished by the various moods of the Syllogism, but it is quite a different matter when we come to handle the complicated results which follow upon the combination of four or five terms. Those who have only looked at the simple diagrams given by Hamilton, Thomson, and most other logicians, in illustration of the Aristotelian Syllogism, have very little conception of the intricate task which would be imposed upon them if they tried, with such resources, to illustrate equations of the type that we must be prepared to take in hand.

As the syllogistic figures are the form of reasoning most familiar to ordinary readers, I will begin with one of these, though they are too simple to serve as effective examples. Take, for instance,

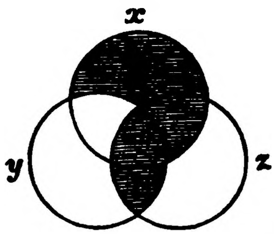
No Y is Z,
All X is Y,
$\therefore$ No X is Z.

This would commonly be exhibited thus:



It is easy enough to do this; for in drawing our circles we have only to attend to two terms at a time, and consequently

the relation of X to Z is readily detected; there is not any of that troublesome interconnexion of a number of terms simultaneously with each other which gives rise to the main perplexity in complicated problems. Accordingly such a simple example as this is not a very good one for illustrating the method now proposed; but, in order to mark the distinction, the figure to represent it is given, thus:



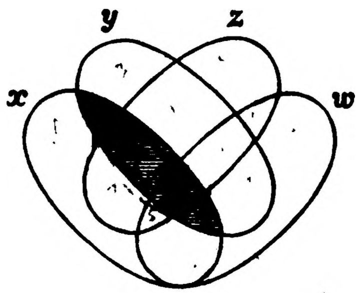
In this case the one particular relation asked for, viz. that of X to Z, it must be admitted, is not made more obvious on our plan than on the old one. The superiority, if any, in such an example must rather be sought in the completeness of the pictorial information in other respects—as, for instance, in the intimation that, of the four kinds of x which originally had to be taken into consideration, one only, viz. the $xy\bar{z}$, or the 'x that is y but is not z', is left surviving. Similarly with the formal possibilities of y and z: the relative number of these, as compared with the resultant actualities permitted by the data, is detected at a glance.

As a more suitable example consider the following—

$$\begin{cases} \text{All } x \text{ is either } y \text{ and } z, \text{ or not } y, \\ \text{If any } xy \text{ is } z, \text{ then it is } w, \\ \text{No } wx \text{ is } yz; \end{cases}$$

and suppose we are asked to exhibit the relation of x and y to each other. The problem is essentially of the same kind as the syllogistic one; but we certainly could not draw the figures in the off-hand way we did there. Since there are four terms, we sketch the appropriate 4-ellipse figure, and

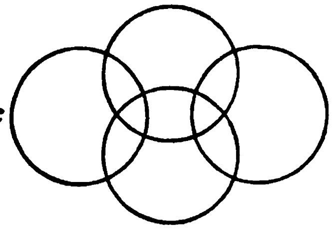
then proceed to analyse the premises in order to see what classes are destroyed by them. The reader will readily see that the first premise annihilates all 'xy which is not z', or $xy\bar{z}$; the second destroys 'xyz which is not w', or $xyz\bar{w}$; and the third 'wx which is yz', or $wxyz$. Shade out these three classes, and we see the resultant figure at once, viz.



It is then evident that *all xy* has been thus made away with; that is, x and y must be mutually exclusive, or, as it would commonly be thrown into propositional form, 'No x is y.'

I will not say that it would be impossible to draw Eulerian circles (or rather closed figures of a more complicated kind, for circles would not do here) to represent all this, just as we draw them to represent the various moods of the syllogism; but it would certainly be an extremely intricate and perplexing task to do so. This is mainly owing to the fact already alluded to, viz. that we cannot break the process up conveniently into a series of easy steps, each of which shall be complete and accurate as far as it goes. But it should be understood that the failure of the older method is simply due to its attempted application to a somewhat more complicated set of data than those for which it was designed; although these data are really of the same kind as when we take the two propositions 'All x is y', 'All y is z', and draw the customary figure. When the problem, however, has been otherwise solved, it is easy enough to draw a figure of the

old-fashioned, or "inclusion-and-exclusion" kind, which shall to a certain extent represent the result, as follows,

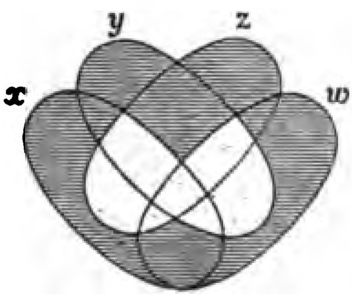


but one may safely assert that not many persons would have seen their way to drawing it at first hand for themselves[1].

One main source of aid which diagrams can afford is worth noticing here. It is that sort of visual aid which is their especial function. Take the following problem:—'Every x is either y or z; every y is either z or w; every z is either w or x; and every w is either x or y: what further condition, if any, is needed in order to ensure that every xy shall be w?' It is readily seen that the first statement abolishes any x that is neither y nor z, and similarly with the others; so that the

[1] Even then we have said more in this figure than we are entitled to say. For instance, we have implied that there *is* some x which is w. The other scheme does not thus commit us; for though the extinction of a class is final, the fact of its being let alone merely spares it conditionally. It holds its life subject to the sentence, it may be, of more premises to come. This must be noticed, as it is an important distinction between the customary plan and the one here proposed. The latter makes the distinction between rejection and non-rejection—such non-rejection being provisional, and not necessarily indicating ultimate acceptance. The former has to make the distinction between rejection and acceptance; for the circles must either intersect or not, and their non-intersection must be construed to indicate the definite abandonment of the class common to both. Hence the practical impossibility of appealing to such diagrams for aid in representing complicated groups of propositions.

four abolished classes are $x\bar{y}\bar{z}$, $y\bar{z}\bar{w}$, $z\bar{w}\bar{x}$, and $w\bar{x}\bar{y}$. Shade them out in our diagram, and it stands thus :—

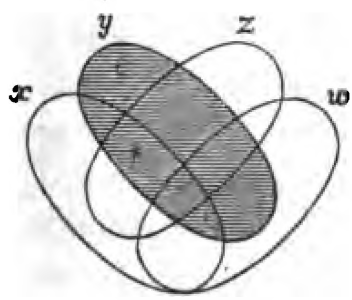


It is then obvious that, of the surviving component parts of xy, one only (viz. $xyz\bar{w}$) is not w. If, then, this be destroyed, *all* xy will be w; that is, the necessary and sufficient further condition is that 'all xyz is w'.

In the same way the implied total abolition of any one class is thus made readily obvious. Take, for example, the following premises: and let us ask quite generally for any obvious conclusion which follows from them :—

Every y is either x and not z, or z and not x;
Every wy is either both x and z, or neither of the two;
All xy is either w or z, and all yz is either x or w.

It will be found on examination that these statements involve respectively the abolition of the following classes, viz. :— (1) of yxz, $y\bar{x}\bar{z}$; (2) of $wyx\bar{z}$ and $wy\bar{x}z$; (3) of $xy\bar{w}\bar{z}$ and $yz\bar{x}\bar{w}$. Shade out the corresponding compartments in the diagram, and it presents the following appearance—



It is then clear at a glance that the collective effect of the given premises is just to deny that there can be any such class of things as y in existence, though they leave every one of the remaining eight combinations perfectly admissible. This, then, is the diagrammatic answer to the proposed question.

It will very likely be objected that we are here taking account only of universal propositions, and that any system of diagrams must be prepared to represent particulars also. We are quite able to meet this demand, though, I must admit, with a certain loss of elegance and simplicity. This seems inevitable; on the ground that, though each of these two forms yields a dichotomy, the consequent alternatives are not of the same nature in each case. Those of the universal consist in destruction and mere non-destruction: those of the particular in preservation and mere non-preservation[1]. Hence, when the two kinds of proposition are combined in the same problem we need three distinct sorts of indication: one for marking the compartments unconditionally destroyed; one for those unconditionally saved; and one for those as to the state of which we are left in doubt.

In another respect the particular proposition leads to somewhat greater intricacy. The destruction of a compartment entails the loss of all its subdivisions, for the negation of a disjunction is distributive; whereas the corresponding affirmation is only alternative. That is, when x is destroyed we know that this involves the fate of every one of its ultimate constituents, such as yz, $y\bar{z}$, &c. But when x is to be saved, all we can say is that some one or more of these ultimate constituents must be secured from destruction, but we have no means of saying off-hand which these must be.

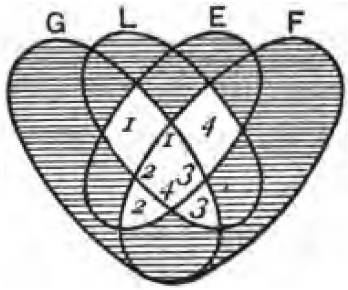
[1] This will be fully explained in the next chapter.

As a matter of fact, I apprehend, the increased intricacy caused by the treatment of particular propositions is no very serious hindrance. We must not be misled by the position which they occupy in ordinary Logic; where the very narrow schedule of admissible propositions, and the requirements of the syllogism, seem to give them a relative frequency and importance far beyond what they really deserve. In what may be called real life, and still more in science, they take a much more humble place. Science aims always at the universal and the definite; and the true logical particular is seldom there regarded as anything more than a temporary correction of some previously accepted or proposed universal, which we are bidden to pause over until we can replace it by some restricted universal which shall be certainly true. When it cannot be thus asserted as true under ascertained conditions, then the aim of science is to assign it numerical definiteness, and so to hand it over to either Statistics or Probability. In ordinary life it is notorious that very many of the propositions to which the logician insists upon prefixing his bare 'some', had really presented themselves with the more quantitative prefixes of 'many' or 'most', and consequently with larger capacities of inference. The true particular proposition has nothing approaching the value, outside the bounds of common Logic, which is assigned to it within them.

Still it is desirable to shew that our diagrams, with a slight modification, can be fitted to deal with propositions of this description. All that is necessary is to introduce a third kind of intimation, in addition to the shading and the leaving blank, hitherto adopted. We might, for instance, introduce two kinds of shading: say, by using vertical lines for the compartments destroyed, and horizontal lines for those saved; whilst the unshaded space con-

tinued to represent the doubts unremoved by the given data. But the alternative character of the information given by the particulars would render it difficult to carry out this plan; and perhaps the simplest way is to use numerals to mark the compartments one or other of which is to be secured by each particular proposition.

Take, for instance, the following example. At a certain examination centre, where either Greek or Latin is a compulsory subject, it is a rule that any one who takes Greek or Latin only must take both English and French; and that any one who takes both Greek and Latin must take either English or French. It is supposed to be known that every combination of three of these languages is, as a matter of fact, taken by some candidate: can we thence infer that there are really any candidates who take up all four?



Draw the diagram, (the initial letters standing for the four subjects,) as usual; the assigned regulations abolish, respectively; $\bar{G}\bar{L}$, $G\bar{L}\bar{E}$, $G\bar{L}\bar{F}$, $\bar{G}L\bar{E}$, $\bar{G}L\bar{F}$, $GL\bar{E}\bar{F}$. Shade these out, including of course what lies outside.

Now the assigned empirical facts establish the existence, respectively, of *GLE*, *GEF*, *GLF*, and *LEF*: mark these by 1, 2, 3, and 4. It is obvious that all these four compartments can be just saved without the necessity of resorting to the central portion. That is, although there *may* be candidates who take up all four subjects, we cannot infer that there *must* be any.

Of course this is complicated, but the complication lies in the nature of things, rather than in the depraved ingenuity of the logician. We shall see (v. Chap. XII.) that with four terms the number of selections of combinations to be dealt with is 2^{2^4}, when the only alternatives were those of taking or not taking. When a third possibility has to be taken into account the numbers increase by powers of 3, those for four terms being 3^{2^4}, or 43,046,721. This is the number of different ways in which the possible combinations of four class terms can be abolished, spared, or left in doubt.

It will be easily seen that such methods as those described in this chapter readily lend themselves to mechanical performance. I have no high estimate myself of the interest or importance of what are sometimes called logical machines, and this on two grounds. In the first place, it is very seldom that intricate logical calculations are practically forced upon us; it is rather we who look about for complicated examples in order to illustrate our rules and methods. In this respect logical calculations stand in marked contrast with those of mathematics, where economical devices of any kind may subserve a really valuable purpose by enabling us to avoid otherwise inevitable labour. Moreover, in the second place, it does not seem to me that any contrivances at present known or likely to be discovered really deserve the name of logical machines. It is but a very small part of the entire process, which goes to form a piece of reasoning, which they are capable of performing. For, if we begin from the beginning, that process would involve four tolerably distinct steps. There is, first, the statement of our data in accurate logical language. This step deserves to be reckoned, since the variations of popular language are so multitudinous, and the terms often so ambiguous, that the data may need careful consideration before they can be reduced to form. Then,

secondly, we have to throw these statements into a form fit for the engine to work with—in this case to reduce each proposition to its elementary denials. It would task the energies of any machine to deal at once with the premises employed even in such simple examples as we have offered, if they were presented to it in their original form. Thirdly, there is the combination or further treatment of our premises after such reduction. Finally, the results have to be interpreted or read off. This last generally gives rise to much opening for skill and sagacity; for though in such examples as the last but one (in which one class, y, was simply abolished) there is but one answer fairly before us, yet in most cases there are many ways of reading off the answer. It then becomes a question of tact and judgment which of these is the simplest and best. For instance, in the example on page 129, there are a number of alternative ways of reading off our conclusion; and until the decision is made between them the problem cannot be said to be solved. I hardly see how any machine can hope to assist us except in the third of these steps; so that it seems very doubtful whether any thing of this sort really deserves the name of a logical engine.

It may also be remarked that when we make appeal, as here, to the aid of diagrams, the additional help to be obtained by resort to any kind of mechanical contrivance is very slight indeed. So little trouble is required to sketch out a fresh diagram for ourselves on each occasion, that it is really not worth while to get a machine to do any part of the work for us. Still as some persons have felt much interest in such attempts, it seemed worth while seeing how the thing could be effected here. There is the more reason for this, since the exact kind of aid afforded by mechanical appliances in reasoning, and the very limited range of such aid, do not seem to be always appreciated.

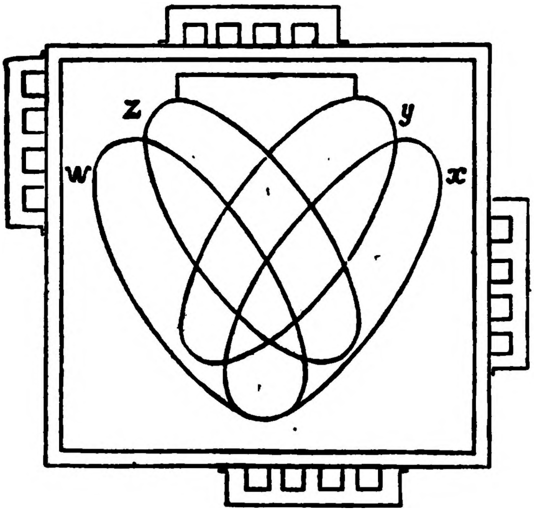
For myself, if I wanted any help in constructing or employing a diagram, I should just have one of the three-, four-, or five-term figures made into a stamp; this would save the few seconds required in drawing them; and we could then proceed to shade out or otherwise mark the requisite compartments. More help than this would be of very little avail. However, since this is not exactly what people understand by a logical machine, I have made two others, in order to give practical proof of feasibility.

For instance, a plan somewhat analogous, I apprehend, to Jevons's *abacus* would be the following:—Have the desired diagram (say the five-term figure with its thirty-two compartments) drawn on paper and then pasted on to thin board. Cut out all the subdivisions by following the lines of the different figures, after the fashion of the children's maps which are put together in pieces. The corresponding step to shading out any compartment would then be the simple removal of the piece in question. We begin with all the pieces arranged together, and then pick out and remove those which represent the non-existent classes. When every one of the given premises has thus had its turn, the pieces left behind will indicate all the remaining combinations of terms which are consistent with the data. I have sometimes found it convenient, where the saving of a little time was an object, to use a contrivance of this kind. There is no reason to give a drawing of it, since any one of the figures we have hitherto employed may really be regarded as such a drawing.

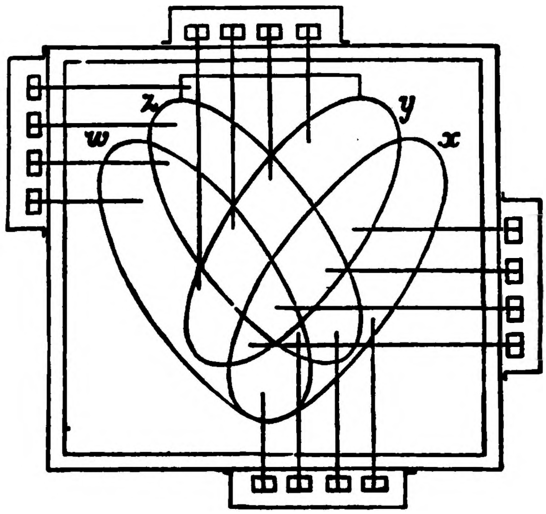
Again, corresponding to Jevons's logical machine, the following contrivance may be described. I prefer to call it merely a logical-diagram machine, for the reasons already given; but I suppose that it would succeed in doing all that can be rationally expected of any logical machine. It is

intended to work for four terms; and the following figures will serve to shew its construction:—

I.



II.



The first figure represents the upper surface of the instrument. It shews the diagram of four ellipses, the small irregular compartment at the top of them being a representative part of the outside of all the four class-figures; that is, this compartment stands for what is neither x, y, z, nor w, or $\bar{x}\bar{y}\bar{z}\bar{w}$. The second figure represents a horizontal section through the middle of the instrument. Each of the ellipses here is, in fact, a section of an elliptical cylinder, these cylinders intersecting one another so as to yield sixteen compartments. Each compartment has a wooden plug half its height, which can move freely up and down in the compartment. When the machine is ready for use each plug stands flush with the surface, being retained there by a pin; we therefore have the appearance presented in fig. 1. When we wish to represent the destruction of any class, all we have to do is slightly to draw out the appropriate pin (the pins of course are duly labelled, and will be found to be conveniently grouped), on which the plug in question drops

to the bottom. This, of course, is equivalent to the shading of a subdivision in the plane diagram. As the plugs have to drop independently of one another, a certain number of them, it will be seen, have to have a slot cut in them, so as to play free from the pins belonging to other plugs. When the plugs have to be returned to their places at the top, all we have to do is to turn the instrument upside down, when they instantly fall back, and on pressing in the pins again they are retained in their place. The guards outside the pins are merely to prevent these from being drawn entirely out.

The only other logical machine of which I have seen any subsequent notice is one by Dr Marquand (described in the *Proceedings of the American Acad. of Arts and Sciences*, 1885, but constructed some years before). It rests upon the same general conception as that of Jevons, but seems to possess some mechanical and technical advantages. The 16 possible combinations of 4 terms are indicated by 16 pointers, which can stand vertical or horizontal to represent negation or non-negation. The premises have, of course, to be interpreted into their resultant denials, before appeal is made to the machine; and the consequences interpreted into a verbal conclusion after the machine has done its work. That is, what the machine does is to make a handy arrangement of the aggregate of constituent denials. It is not easy, indeed, to conceive how any mechanical device could do more than this intermediate work between the original premises and the verbal conclusion.

The above seems all that is necessary to say about the nature and employment of logical diagrams. Their use under any circumstances has been objected against by some purists, but not on grounds with which readers of such a work as this are likely to sympathize. Thus Mansel

declares against them altogether (*Prolegomena*, p. 55), on the ground that they are an attempt to visualize those concepts which while they remain such must be necessarily incognizable by any faculty of sense, and solely accessible to the understanding. Those who adhere to the material view of Logic will of course be but little influenced by such an objection. What we are concerned with is classes of objects, actual or possible, and these may very fairly be represented by circles or other closed figures. Such figures must necessarily include or exclude any part of the extension, just as the class must, and by shading or otherwise marking the figure we can duly indicate whether or not such a class must be pronounced actually non-existent. And this is all that can possibly be expected of any such figure.

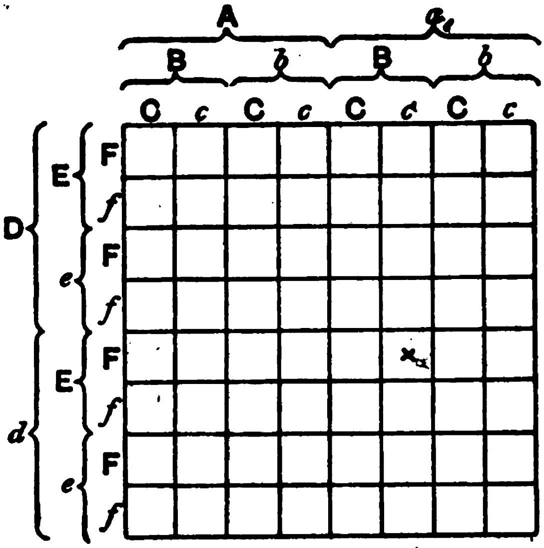
That letters, or some such symbols, are the truly appropriate instruments of calculation must of course be maintained. They alone can be strictly called symbols, and we must be capable of carrying out every process by means of them. But I shall make frequent appeal to diagrams also; both for purposes of mere illustration, and because they will occasionally afford much briefer modes of proof. The objection to them is not that they are in any way inappropriate to Logic, but that like other kinds of pictorial language they are rather too cumbrous for general purposes. If I may judge from my own experience, the case in which an appeal to a diagram results in the most marked economy of labour is that in which we have to compare two complicated expressions, or the outcome of two sets of premises. The reader will find an illustration of what I mean, towards the conclusion of Chap. XVI., in which different solutions of a problem are compared, in order to shew their equivalence. If the expressions, in such cases, are really identical, this can very readily be proved: if they are not, we can with equal

ease localize the discrepancy. In general, this is a much less irksome test than any other.

Of course we must positively insist that our diagrammatic scheme and our purely symbolic scheme shall be in complete correspondence and harmony with each other. The main defect of the common or Eulerian diagrams is that such correspondence is not secured. In fact, as was shown in the first Chapter, those diagrams not only do not fit in with the ordinary scheme of propositions which they are employed to illustrate, but do not seem to have any widely recognized scheme of propositions to which they could be consistently affiliated. The forms of common language are of course peculiar to no system whatever, and therefore can be made to work in with any possible logical scheme, though they adapt themselves to some, such as ours, less than to others. But symbolic and diagrammatic systems are to some extent artificial, and they ought therefore to be so constructed as to work in perfect harmony together. This merit, so far as it goes, seems at any rate secured on the plan above described.

It need hardly be said that what we have been considering in this chapter is a scheme of diagrams, in the ordinary sense of the term: something geometrical in fact, where inclusion and exclusion should be obviously represented by the figures. Jevons' plan, as is well known, was that of writing down all the possible combinations, and striking out those inconsistent with the premises. This method admits of much improvement by judicious arrangement. The most compendious form I have seen is by Dr Marquand, of Johns Hopkins University, who has produced something between what would be commonly regarded as symbolic and diagrammatic.

The following is such a scheme for 6 terms:



Capitals and small letters of course stand for terms and their contradictories, and every possible combination is thus represented: *e.g.* the square marked by a cross is *aBcdEF*. The scheme is very compendious: thus one adapted for 10 terms, and involving 1024 combinations, can be conveniently printed on one of these pages. Of course there is not the help to the eye here, afforded by keeping all the subdivisions of a single class within one boundary; for though this is secured for *A* and *D*, it is lost for *B*, *C*, *E*, and *F*. But this is almost inevitable where we deal with many class terms. If we had occasion to work out problems involving 8 or 10 terms, such a scheme would be a most valuable resource; and we shall, in fact, from time to time have occasion to resort to it in simpler cases than this.

CHAPTER VI.

ON THE IMPORT OF PROPOSITIONS, AS REGARDS THE ACTUAL OR CONVENTIONAL EXISTENCE OF THEIR SUBJECTS AND PREDICATES.

A CHAPTER must be devoted here to the discussion of a connected group of topics, which, though assuming decided prominence in the Symbolic Logic, are by no means properly confined to it. They should, in fact, have been so thoroughly treated elsewhere, and decided one way or the other, that a mere reference here would have sufficed instead of a somewhat elaborate explanation and justification of the view to be adopted. Many logicians, if not a majority of them, have however passed the subject by entirely, interesting and important as it is from any speculative point of view. This is probably owing to the prevalent acceptance until lately of the Conceptualist theory of Logic, to which, at least when rigidly adhered to, such topics as we now propose to enter on are unfortunately somewhat alien.

The simplest way perhaps of introducing the subject now to be discussed is by raising the question whether, when we utter the proposition 'All X is Y', we either assert or imply that there are such things as X or Y, that is, that such things exist in some sense or other? Or again, when we state that 'Some X is Y', does this particular proposition

make any different implication as to this special point of there being any *X* or *Y*?

A word of explanation at the outset, in order to avoid misapprehension. Any discussion about the 'existence' of such and such things will create an impression in some minds that we propose to enter on some kind of metaphysical enquiry. It must be clearly understood therefore that we intend to discuss the question entirely on scientific or logical ground, without digression toward considerations which are more appropriate to metaphysics. As to the nature of this existence, or what may really be meant by it, we have hardly any need to trouble ourselves, for almost any possible sense in which the logician can understand it will involve precisely the same difficulties and call for the same solution of them. We may leave it to any one to define the existence as he pleases, but when he has done this it will always be reasonable to enquire whether there is anything existing corresponding to the *X* or *Y* which constitute our subject and predicate. There can in fact be no fixed tests for this existence, for it will vary widely according to the nature of the subject-matter with which we are concerned in our reasonings. For instance, we may happen to be speaking of ordinary phenomenal existence, and at the time present; by the distinction in question is then meant nothing more and nothing deeper than what is meant by saying that there are such things as antelopes and elephants in existence, but not such things as unicorns or mastodons. If again we are referring to the sum-total of all that is conceivable, whether real or imaginary, then we should mean what is meant by saying that everything must be regarded as existent which does not involve a contradiction in terms, and nothing which does. Or if we were concerned with Wonderland and its occupants we need not go deeper down than they

do who tell us that March hares exist there. In other words, the interpretation of the distinction will vary very widely in different cases, and consequently the tests by which it would have in the last resort to be verified[1]; but it must always exist as a real distinction, and there is a sufficient identity of sense and application pervading its various significations to enable us to talk of it in common terms. No logician who utters a proposition of the form 'All *X* is *Y*', can reasonably refuse to say *Yes* or *No* to the question, Do you thereby imply that there is any *X* and *Y*? And even if the question were evaded in this particular form, the distinction involved in it cannot be evaded. It is at least clear that such a proposition puts '*X* which is *not Y*' upon a different footing from '*X* which *is Y*'. And so with other propositions. It is impossible intelligently to accept a categorical assertion without admitting that a distinction is thereby introduced in respect of the subject-matter with which it deals. Phrase it how we will:—that such and such things do not exist, are denied, &c.: that such and such terms have nothing corresponding to them, &c.—the disjunction in respect of admissibility is necessarily introduced. It may take the form of an alternative between rejection and non-rejection,—as I hold to be the case in universal propositions. It may take the form of an alternative between acceptance and non-acceptance,—as I hold to be the case in particulars. But the same substantial distinction, in some form or other, is unavoidable in all rational assertion and denial. And this is all which we need claim at present.

[1] As Prof. W. James happily puts it, when discussing a somewhat similar topic, "It thus comes about that we can say such things as that Ivanhoe did not *really* marry Rebecca, as Thackeray *falsely* makes him do. The real Ivanhoe-world is the one which Scott wrote down for us. *In that world* Ivanhoe does not marry Rebecca." (*Mind*, XIV. 329.)

The clearest way perhaps of stating the line of discussion here adopted is by claiming the two following postulates. I state them explicitly and call attention to them because they are not familiar to logicians, if indeed they have ever been definitely enunciated.

(1) That we must be supposed to know the nature and limits of the universe of discourse with which we are concerned, whether we state them or not. If we are talking of ordinary phenomena we must know whether we refer to them without limit of time and space; and if not, within what limits, broadly speaking. If we include the realms of fiction and imagination we must know what boundaries we mean to put upon these.

(2) That we must come furnished with some criterion of existence and reality suitable to that universe. That is, all our assertion and denial must admit, actually or conceivably, of *verification*. Put the propositions 'All X is Y', 'No X is Y', into the forms $X\bar{Y}=0$, $XY=0$, and the statements 'There is no $X\bar{Y}$', 'There is no XY', must admit of verification and be intelligible, without any necessary digression into metaphysics.

Take the following comparison by way of illustration. The question whether or not a certain statement has been published clearly requires us to know exactly what books are to be appealed to, as decisive in the matter. But it is equally clear that the point can be settled whatever the books thus chosen may be. The ordinary stature of man could be decided as unequivocally by appeal to Swift as to Quetelet if we have made up our minds what are the 'men' we are talking of. The verification of a statement can be carried out by appeal to a novel or a scientific work, to a fairy tale or a blue-book, provided we have decided what our authorities are to be.

Here, as on several other occasions in this work, we are in face of three very distinct questions which we must prevent from getting entangled more than is absolutely unavoidable. The first is one of popular convention; the second is one of ordinary logical legislation and usage; and the third is one of convenience and consistency in the working out of the Symbolic or Generalized Logic. The first two of these do not much concern us here, for we have had to repudiate once for all any bounden obligation to either the language of common life, or that of the common logic. They must therefore be passed over much more slightly than they deserve. It is however impossible to neglect them altogether, since it is only by realizing the difficulties which meet us in both of those quarters that we can appreciate the weight of argument in favour of the conclusion about to be stated.

I. The first question then is merely one of custom or usage: what do ordinary persons think and understand on this point? It is quite impossible to answer this question offhand; partly owing to the immense number and variety of the popular forms of assertion, and partly to the multitudinous associations contained in them, these varying from the barest suggestion up to implication scarcely short of direct assertion. To the best of my judgment we should have to decide somewhat to the following effect.

The Universal Affirmative. Broadly speaking, the statement that All X is Y does imply directly that there are X's, and consequently indirectly that there are Y's. If any non-mathematician were told that all rectangular hyperbolas have their asymptotes at right angles to one another, he would assume unhesitatingly that there are such things as rectangular hyperbolas, that they exist in the domain of mathematics. And so with most other of the things about which we have occasion to make affirmative assertions. The

main ground for this assumption seems to be the very obvious one that the practical exigencies of life confine most of our discussions to what does exist, rather than to what might, or once did, but does not now; and that here as elsewhere, where one thing is the rule and another the exception the *primâ facie* presumption is in favour of the former. People do not in general talk about what they believe to be nonentities. They do not, for instance, without warning describe the qualifications for an office which they suppose does not exist, nor do they state the attributes of a substance which is nowhere, within the scope of their enquiry, to be found. In saying this the reader must keep in mind that we are not discussing the question whether every separate word, that is, substantive or adjective, has things corresponding to it. The question is whether single words, or groups of words, when forming subjects and predicates of propositions, are to be understood to make this implication within their then actual range of application.

This seems to be the rule; but there are some admitted classes of exceptions. For instance, assertions about the *future* do not carry any such positive presumption with them, though the logician would commonly throw them into precisely the same 'All X is Y' type of categorical assertion. 'Those who pass this examination are lucky men' would certainly be tacitly supplemented by the clause 'if any such there be'. So too, in most circumstances of our ordinary life, wherever we are clearly talking of an ideal. 'Perfectly conscientious men think but little of law and rule', has its signification without implying that there are any such men to be found. Other similar cases will probably occur to the reader, but the broad conclusion remains that when we are speaking of facts within our power to verify we do not without warning make predications of non-existent

subjects. And the existence of the subject of the proposition being thus established, it follows clearly that that of the predicate must be so too.

The Particular Affirmative and Negative. The same assumption seems to rule here, but in a more unqualified manner, owing to the fact that most of the exceptions admitted there could have no place here. An assertion confined to 'some' of a class generally rests upon observation or testimony rather than on reasoning or imagination, and therefore almost necessarily postulates existent data, though the nature of this observation and consequent existence is, as already remarked, a perfectly open question. 'Some twining plants turn from left to right', 'Some griffins have long claws', both imply that we have looked in the right quarters to assure ourselves of the fact. In one case I may have made my observations in my own garden, and in the other on crests or in the works of the poets, but according to the appropriate tests of verification, we are in each case talking of what *is*.

The Universal Negative. So far as I can judge, it seems that we very commonly make the same assumption as before in regard to the subject, but do not feel equally confident as regards the predicate. 'No substance possesses a temperature below − 280° centigrade' (I must again remind the reader that we are considering the subjects and predicates as wholes, not the separate elements of which they may be composed). Since only substances are supposed to possess a temperature at all, this negatives the existence of the predicate altogether. The exceptions here seem of much the same kind as in the case of the Affirmative, and perhaps more frequent; but owing to the general reluctance of men to quit the ground of fact altogether, I presume that where the subject does not exist we should generally find that the predicate does:

e.g. 'No perfectly wicked character is to be found in fiction'. As an instance of a possibly non-existent subject of a negative proposition, take the following:—'No person condemned for witchcraft in the reign of Queen Anne, was executed'. I have verified this, let us suppose, by searching all the records of executions within the time specified. There would surely be no impropriety in my publishing this fact before ascertaining, by resort to records of trials, that there really had been any persons so condemned. The disproval of this fact, which would be equivalent to showing that the subject of assertion had no existence, would at most show that I had been hasty. It would not make the proposition itself invalid. Where the sentence can stand equally well in the converted form it seems that we decidedly prefer that the subject, as distinguished from the predicate, should be a reality. Thus the two propositions 'No unpardonable sins are sins', and 'No sins are unpardonable sins', are logically equivalent; one being the simple converse of the other. The objection would of course be raised in each case that what is really meant is that there are no such things as unpardonable sins, and that it would therefore be best to say so at once; but whereas the former proposition strikes every one as absurdly stated, the latter is at most awkwardly stated. The ground of this distinction seems to me to lie in the natural dislike to a non-real subject of such a proposition, where this can be avoided.

Any account of popular phraseology which is thus confined to the accepted four fundamental forms of logical predication is of course very imperfect, but we have no space to enquire as to the import of other forms. They seem to carry the most various degrees of implication with them. For instance, 'None but X are Y':—No one on hearing this would be surprised if the speaker went on to say 'And I do not believe there are any X's'. But the logician

would commonly phrase this statement as either 'All Y are X', with the consequent implication that there are Y's and X's, or, 'No not-X are Y', with the consequent implication that there are not-X's and Y's. This does not seem to be quite what the ordinary mind contemplates as implied in such a phrase.

II. We must now see what the logicians have to say upon the matter. The most direct way of deciding this point would be to examine their assertions, but here unfortunately we shall obtain very little help indeed. Most of the treatises written by English authors, if I am not mistaken, entirely omit all reference to the subject; at least I can find no sort of critical examination of it[1]. There are

[1] In several English treatises the question is raised as to whether a *class-term* postulates the existence of things corresponding to it, *i.e.* whether the class must be actual or merely potential. But this question, though doubtless connected with the one before us, is quite distinct. All that here concerns us is whether a *whole subject or predicate*, as such, consisting perhaps of a complex of terms, demands such a postulate.

Even those who have more nearly approached our present point of view have done so under two limitations which render most of their discussion irrelevant to our purpose. For (1) they have only (so far as I see) touched upon the case of universal affirmative propositions. And (2) they have mostly assumed that the contrast demanded was always the same, viz. that between phenomenal or sensible existence on the one hand and the region of the imaginary on the other (see, for instance, Mill's *Logic*, I. ch. VIII, where he discusses the dragon and its flame-breathing habits). I want to make the question perfectly general. Take what test we please of existence, and what universe of discourse we please, and we ought to be prepared, when A and B are anyhow connected in a proposition, at least to face the question whether there 'is' or 'is not' an A or a B or an AB. No one who considers such a question as beside the mark can be considered to assert or deny intelligently.

Even Mansel seems to admit this much when he says (*Prolegomena*, p. 67) "When I assert that A is B,...[I mean] that the object in which the one set of attributes is found is the same as that in which the other set is found."

Of English writers De Morgan is the only one who has entered fully on the subject, deciding in favour of

indeed two distinct opinions which have been advanced more or less by way of implication, which as representing to my thinking each of them a reduction to absurdity, but in opposite directions, deserve a short notice.

For instance, it is sometimes stated or implied that subjects and predicates have no reference whatever to the phenomenal existence or non-existence of things corresponding to them. This is presumably the view of the rigid conceptualists; at least it is not easy to put any other construction upon much that they say. Mansel repeatedly maintains that we are concerned with *concepts* only and their mutual relations to one another. On this view it might be urged that we are uniformly certain of the existence of the idea or concept in our own minds, and uniformly uncertain (from a logical point of view) of any phenomena corresponding to it. It seems impossible to carry out this view consistently. We may, by a shift, adhere to it in the case of propositions dealing with essential properties;—thus when we say that 'all honest men are deserving of respect' we may maintain that we are thinking merely of groups of attributes present to our mind, and that 'deserving of respect' is found amongst those which compose the characteristics of 'honest men'. But how can we adhere to this in the case of accidental attributes, which hardly any one ever thought of putting into the subject till he heard

the implication of existence (*Formal Logic*, p. 111. *Syllabus*, p. 10. *Eng. Cyclopædia*, Logic). Herbart is the best known philosopher perhaps who has supported the opposite view (*Einleitung*, § 53), in which he is followed by Sigwart and some others. See also Beneke (*Logik*), Ueberweg (*Logik*, §§ 68, 69, 85) and C. S. Peirce (*Journal of Spec. Phil.* 1868, p. 140). But since these writers have (generally speaking) expressed their views subject to the limitations indicated above, I cannot with certainty claim them as for or against my own view. (The whole subject has been since well discussed by Dr Keynes in his *Formal Logic*.)

the assertion made? So too with particular propositions. These certainly imply actual observed or recorded occurrences[1].

Even if we insist upon turning everything about us into a concept when we have to deal with general propositions, we shall not find it so easy to carry this out elsewhere, for Proper names and Individual propositions have a good deal of vitality about them which resists this process of conversion into the state of a concept. And if, with Mansel, we are resolutely consistent in making no exception here, what are we to say to Existential propositions? We are always liable to encounter these, and terms whose essence (in old language) is to exist. They will still persist, in spite of our efforts to make away with them, and so stalk about with the glaring incongruity of a living body amongst the world of shades composed of our concepts.

It may be remarked that this question as to the import of propositions is intimately connected with that of Hypotheticals. As all the more consistent logicians who have adopted the view now under notice have recognized (*e.g.* Herbart, Beneke, Sigwart) we thus do away with any general distinction between the hypothetical and the categorical. That this must be so can be illustrated by a very simple example:—'If there are any *X* which are *Y* then they are *Z*', this is clearly, in the opinion of most logicians, hypothetical. But it may be phrased, without the slightest change of meaning, 'All *XY*, if such there be, are *Z*'; and this is identical in form with the ordinary categorical 'All *A* is *B*' when we accept it with the tacit proviso 'All *A*, if there be any'.

This rejection of the Hypothetical, as a distinct form, will not be any sacrifice to us, for it is (as will be fully shown in

[1] See a fuller discussion of this subject in chap. XIX.

a future chapter) a marked characteristic of this generalized Logic to weaken that distinction. But it is important as showing the really serious nature of the difficulty from the point of view of the ordinary Logic. How those writers who adopt the hypothetical interpretation explicitly[1], or those who adopt it implicitly by enunciating a theory[2] which consistently implies it, can still retain the distinction between Categorical and Hypothetical as anything more than an optional one of expression, is to me unintelligible.

Another view, equally extreme in the opposite direction, has been maintained by Jevons; viz. that *every* term, as well as its contradictory, must alike be claimed as represented in fact, at least when they occur as subject or predicate of a proposition[3]. It is hard to account for such a view on any system, but especially on that of Jevons; where so far from its being a consistent deduction from his principles it seems a merely capricious restriction. It is not as if complex terms differed from simple terms in respect of either their signification or their application. If we have to deal throughout a piece of reasoning with such a term as AB we should not hesitate to substitute a single symbol for it, any more than we should hesitate to express by a single symbol such a class as 'tory publicans' if we had to predicate anything of them.

Most logicians, I apprehend, entertain an intermediate view, more nearly in accordance with the conclusions of common sense and common language as indicated above.

[1] *e.g.* Spalding (*Logic*, p. 104) "X is Y, has logically no more meaning than this: if X is and if Y is, then X is Y."

[2] I refer of course to the thorough going Conceptualists. Mansel, indeed, is more consistent, rejecting the distinction of hypothetical by maintaining that "All formal thinking is, as regards the material character of its objects, problematical only" (Aldrich, p. 236).

[3] See Jevons's *Pure Logic*, p. 65.

In any complete treatise on Formal Logic it would be necessary to enter into this question somewhat fully, so as to ascertain in what direction the weight of judgment and reason must be considered to lie. Discussion however of this kind being alien to the object of this volume it will be better to approach the subject in a somewhat different way; namely, by enquiring what can be elicited from the universally accepted rules of Logic. This will also have the advantage of indicating those difficulties in the subject, the desirability of avoiding which has induced me to accept the doctrine to be presently explained.

Perhaps the most conclusive evidence from this source is to be found in the accepted rules of the syllogism. For instance, 'All *Y* is *Z*' 'All *Y* is *X*', would not enable us to conclude as we do, 'Therefore some *X* is *Z*', if the propositions had to be interpreted 'All *Y*, if there is any, is *Z*'. They must certainly be interpreted 'All *Y*, *and there is such*, is *Z*'. Since however no special assumption is announced in the case of *Darapti* we must fairly conclude that *all* universal affirmatives postulate the presence, so to say, of actual representatives of their subjects, and consequently of their predicates[1].

This is all very well to begin with, but observe to what length it will lead us if we accept also other commonly received rules. For instance, are we to be allowed to

[1] Is the following argument fallacious? 'All tory publicans are tory: All tory publicans are publicans: Therefore (*Darapti*) some publicans are tory'. If this is formally correct, then it would seem that all those who hold, with Jevons, that the question of existence, as applied to our terms, has no application in Logic, as also all those who do not, with Spalding, introduce the tacit addition to the subject of a proposition "if such there be", will here have a hint for a general *à priori* proof that 'Some *S* is *P*', whatever *S* and *P* may be. (See Keynes' *Formal Logic*, Ed. II. p. 324.)

contraposit propositions? If so we get at once into implications about negative terms. From 'All X is Y' we are commonly allowed to derive 'All not-Y is not-X'. But this being a universal affirmative must indicate that there are instances of not-Y and not-X, as well as of Y and X. This is certainly very remote from the popular view, which never thinks of insisting that X and Y must not only exist but must also abstain from comprising all existence. The popular interpretation of this form of proposition is surely conditional only, and equivalent to 'All not-Y, if there be any, is not-X'. Then again as regards negative propositions. From 'No X is Y', we infer without hesitation, 'All X is not-Y'. Consequently the *negative* proposition also must refuse merely possible subjects and claim them as existent[1]. And since the universal negative is simply convertible, what holds of its subject must also hold of its predicate. Here again we are at variance with the popular view, which seems to recognize a distinction in this respect between the subject and the predicate of these propositions.

It really seems then as if the commonly accepted rules of Logic, when pressed to their conclusions, would force us into

[1] These remarks apply mostly to the handling of modern logicians. It is well known that the older logicians rather avoided these negative subjects and predicates, and that the rules of the syllogism were not devised for their employment. For instance several logicians have noticed that the rules against drawing a conclusion from two negative premises were only valid under the supposition that we might not thus introduce a fourth term under the guise of a negative term. Thus both Ploucquet (*Sammlung*, p. 78) and Darjes (*Weg zur Wahrheit*, p. 227) have given various instances which may be expressed by the letters 'No A is B', 'No A is C', and have shown that if we were allowed to phrase them in the form 'All A is not-B', 'All A is not-C', we might draw the conclusion 'Some not-C is not-B'. But knowing under what restrictions the common rules were drawn up, they did not therefore charge these rules with being incorrect.

that extreme view, noticed above, as being held by Jevons. But none the less that view seems to me a reduction to absurdity. It involves not only a needless departure from all popular convention and association, but, if adhered to, it would seriously hamper us in all our logical predication. Are we never to assert anything or deny anything about X or Y, unless we are certain not only that there are things which are X and Y, but also things which are not X, and not Y? The former condition might not be any great hardship, but the latter would extremely curtail our customary rights of denial. Rather than submit to such restraint we should prefer to abandon the claim to contraposit our propositions, or in any way to extract negatives out of positives. Probably also we should think it best not to hold out for the right to simply convert a universal negative proposition. Otherwise we should hardly have elbow-room left in which to assert or deny with any sense of freedom.

Such difficulties as those just mentioned encounter us within the field of ordinary Logic, and amongst the simple propositions with which it commonly has to deal, but they would be but the beginning of our troubles in the wider field of Symbolic Logic. We there have to deal with a number of propositions simultaneously, involving perhaps half-a-dozen or more of terms, several of these terms being sometimes combined into a single subject or predicate[1]. We should soon find that it was simply

[1] Professor Bowen (*Logic*, p. 144) has seen the difficulty in the case of these complicated subjects and predicates. He makes a special exemption in the case of, say, XY, as a subject; terming the proposition a 'limitative' one, and reading the subject 'X if it be Y'. Subjects so compounded he considers do not postulate any existence, but should take their place amongst hypotheticals. He thus escapes the difficulty involved in the example on page 156. As stated above, a generalized system of logic cannot recognize any distinction between one and the other of

impossible to adhere to a system in which either categorical assertion or denial were permitted to carry along with it the implication of the existence of things corresponding to the subject or predicate. Any group of such propositions would form a very precarious structure, for it might be discovered on analysis that taken together they demanded the annihilation of one or other of the subjects or predicates involved in them. The discovery of such a result would then require the rejection of the whole group as somehow involving mutual inconsistencies; but the solution of the difficulty would not be furnished by the data.

For instance, take the group,

$$\begin{cases} \text{All } x \text{ is either } y \text{ and } z\text{, or not } y, \\ \text{All } xy \text{ is } w \text{ or not } z, \\ \text{No } wx \text{ is } yz. \end{cases}$$

There is certainly nothing intrinsically amiss with any one of these propositions, and on the rational and simple explanation to be presently offered they will harmonize together perfectly. But taken together they demand, and are satisfied by the condition 'No x is y', or 'There is no such thing as xy'. But xy is the subject of one of them, and this fact would therefore (if we admit the implication now under discussion) put a veto upon such a conclusion. In that case we should have nothing else to do than to dismiss the whole group of propositions together as involving an inconsistency somewhere, though we could not tell where; for we should have no right to assign the fault of the inconsistency to one of the propositions rather than to another.

Finally, as a climax to all this, we must remember that

these forms of proposition; though I think Mr Bowen is quite right in considering that the conventions of popular speech do put them upon a distinct footing.

the scope of our propositions has to be limited to a certain universe of discourse. This consideration immensely aggravates the burden of all these restrictions; for we should thus be forced in consistency to ascertain, not only that there are X's and Y's somewhere or other, but also that these as well as their contradictories prevail within the sphere, perhaps a very narrow one, which constitutes the universe of our discourse for the time being. For it is certainly to this only that our premises apply, and in reference to which their import must be interpreted.

III. There is however another view open to us which will, I think, avoid all the difficulties above mentioned, together with others which will present themselves in the course of our discussion. Or perhaps one ought rather to say that we may succeed in this way in removing all such portion of the attendant difficulty as arises from real uncertainty of meaning, for though this comprises much the larger part of our trouble it certainly does not exhaust it. The view now to be explained is, I think, almost necessarily forced upon us by the study of Symbolic Logic, though when it has been once realized we should find that it could be applied also to the ordinary logical interpretation of the proposition:—whether it is expedient to insist upon applying it there, is, of course, a question upon which we cannot here enter.

Every universal proposition may of course be put into a negative form: this is familiar to every logician, the distinction of our system being that this negative side of a proposition is more consistently and uniformly developed, and provided with a suitable symbolic notation. Now if we adopt the simple explanation (of a universal proposition) that *the burden of implication of existence is shifted from the affirmative to the negative form;* that is, that it is not the

existence of the subject or the predicate (in affirmation) which is implied, but the non-existence of any subject which does *not* possess the predicate, we shall find that nearly all difficulty vanishes[1]. This may need a little explanation and justification. Take the proposition 'All x is y'. There being two class terms here, there are, as has been so abundantly explained, four ultimate classes, viz. xy, x not-y, y not-x, and not-x not-y. Now what we shall understand the proposition 'All x is y' to do is, not to assure us as to any one of these classes (for instance xy) being *occupied*, but to assure us of one of them (viz. x not-y) being *unoccupied*. That is, instead of regarding the *affirmative* form as being the appropriate and unambiguous form, we regard the corresponding or equivalent *negative* form as possessing these attributes. Whether there be any x's or y's we do not know for certain, but we do feel quite sure that there is no such thing existing as 'x which is not y'.

The example on p. 156 showed the necessity of some such understanding as this. A solitary proposition may make positive implications without any mischief, but if it is to be combined with a number of other propositions it must lay aside such claims, for they will very possibly lead to direct mutual conflict. On the other hand, if it confine itself to its negative implication, it will make a contribution to the net knowledge furnished by the other propositions, such as can be set aside by nothing short of a direct contradiction in terms.

[1] It may be pointed out here, once for all, that I am not proposing to lay it down as a law that no proposition can have these positive implications, but merely to maintain that in the Symbolic Logic, as a means to generalization, they should not be regarded as any part of the proposition. Admit them by all means wherever desired; but as explicit implications, to be distinctly indicated.

We are led to the following result therefore; that in respect of what such a proposition affirms it can only be regarded as conditional, but that in respect of what it denies it may be regarded as absolute. The proposition 'All x is y' may be written, and for the purposes in hand is better written, 'No x is not-y' ($x(1-y)=0$, or, more briefly $x\bar{y}=0$, as will be more fully explained hereafter) that is, it empties the compartment $x\bar{y}$, or declares the non-existence of a certain combination, viz. of things which are the combination of x and not-y. But it does not tell us whether there is any y at all; or, if there be, whether there is also any x, or whether the two together make up the total of the things with which we are concerned. All these facts it leaves quite uncertain, carefully limiting itself to the single negation of there being any $x\bar{y}$. Now we know that our symbols are subject to the universal condition represented by $1 = xy + x\bar{y} + \bar{x}y + \bar{x}\bar{y}$; or, put into words, that everything must be either xy, or x not-y, or y not-x, or not-x not-y. When therefore we have thus expunged one of these as not existing, we have three alternatives left: and we know that some one, or some two, or all three of these must be represented amongst existences, but we do not know which of these possible cases may hold good.

The import therefore, as regards existence, of the Universal Affirmative is very plain. Whatever it does in the way of predication, in the way of implication of existence it just denies one combination and no more, leaving the other combinations untouched. Now let us couple another proposition to the former, by asserting that 'All y is x', as well as 'All x is y.' We see at once what results. The second proposition empties out the class $\bar{x}y$, as the former did $x\bar{y}$; that is, it declares the non-existence also of things which are y and not-x. Before, the positive possibilities were three in

number, now they are reduced to two; for it is implied that everything must be either both x and y or neither of the two. Carrying this process one step further, we see that three such propositions would be requisite to establish unequivocally the existence of any one of the four classes. If we expunge $\bar{x}\bar{y}$ also, we are then reduced at last to an assertion of existence, for we have now declared that xy is *all*, viz. that within the sphere of our discussion everything is both x and y.

The positive assertion in this last case is of course of a peculiar kind. We are not merely told that a certain class is represented, but that it is represented to the exclusion of all else, being the sole survivor after all else has been expunged. It may therefore be very fairly asked whether there is no simple form for declaring that such and such a class *is* represented, and for declaring nothing more? There is such a form, but it will have to be sought in the *particular* proposition, and not in the universal. I do not mean that this significance is naturally understood to be involved in such propositions, but that they will readily lend themselves to such an interpretation with very little violence, and are the only ones that will do so. This will be fully explained in the next chapter, but in order to round off the present discussion I will give a few lines of indication of what is meant. The formula $xy = 0$ expresses, as we have seen, that the class in question is absent; whilst $xy = 1$ expresses that it is not only present, but present to the exclusion of all else. Now invent an intermediate form $xy > 0$, which, in accordance with mathematical usage, is to be read off as 'xy does *not* equal 0', that is, is something more than zero. The most natural and simple way of interpreting such a proposition would be 'xy is something', corresponding to the other forms 'xy is all' and 'xy is nothing'; or, still more

simply, 'there is xy.' This seems to me to be very nearly the signification of the familiar form 'Some x is y' of our ordinary Logic, with the important difference, however, that it is free from all ambiguity as to the fact of there being any x and y, for it is expressly understood to signify that this is so[1].

When therefore we meet with the proposition 'all x is y' I shall understand it to be interpreted as follows: (1) negatively and absolutely, 'there are no such things as $x\bar{y}$', or (2) positively but conditionally, 'If there are such things as x, then all the x's are y'. And owing to the superior brevity and absoluteness of the negative form, we shall make it a common practice to write down such a proposition symbolically in the form $x\bar{y} = 0$.

The foregoing results may be summed up by saying that whereas it is quite certain that common usage does make various assumptions as to the existence of the subject and the predicate separately, in Universal propositions, and quite certain that the Symbolic Logic (as here interpreted) does not, the state of the law on this point in the common Logic is full of perplexity.

Much of the above discussion will perhaps appear rather dry and far-fetched, but we shall soon find that it will lead to some interesting and unexpected applications. Begin by examining the mutual relations of two inconsistent pro-

[1] The impossibility of taking account of both universal and particular propositions with such resources as were provided by Boole,—and the same remark applies equally to those of Jevons,—was noticed at an early period. Thus Mr A. J. Ellis pointed out that "an algebra of 0 and 1 can correspond only to a logic of *none* and *all*" (*Proc. of Roy. Soc.* Vol. XXI. 497). I insisted upon this fact in the first edition:—the above paragraph having been but slightly altered. The necessity of some other symbol, to be introduced either into the copula or the predicate, is now universally recognized. The nature of these symbols is fully explained in the next chapter.

positions of the customary kind. Take the two following, which in technical language are termed 'contrary' the one to the other: 'All x is y', and 'No x is y'. It will sound rather oddly at variance with ordinary associations to ask if these two propositions are compatible with each other, that is, if they can both be admitted simultaneously? The reply of ordinary Logic would be an emphatic negative; and this reply would be valid enough from the predicative point of view, provided we look no further than that. But if we take a wider, and I should say a sounder view, we shall readily see that there is no reason whatever against our accepting both these propositions. Look at them from the class or compartment point of view, and we see at once that 'All x is y' simply empties out $x\bar{y}$, whilst 'No x is y' empties out xy. There is no harm in this, no suggestion of conflict or inconsistency. These two negations (for as such we regard them both) when combined together are simply tantamount to denying the existence of x altogether,—an existence which no one had been supposed to assert;—that is, the always implied hypothesis 'x, if x exist' is here decided in the negative.

To the mathematician, or to any one trained in a mathematical way of looking at things, the admission of such a result as this would not even put any strain upon the language which he is in the habit of using. Nothing is more familiar to him than the practice of dealing with some unknown quantity during a continued process of reasoning, and then finding as a conclusion that this unknown quantity (say a root of an equation) is equal to nothing:—that is, not that it must at last be *made* nothing, but that we must now recognize that that was the value which belonged to it all along, under the given conditions of the problem. This possibility of destruction is what his symbols of magnitude

must always be prepared to face as something which may at any moment be declared to be their lot. The only caution which we have to keep in view is that of looking back to see that we have not unconsciously transgressed certain rules (*e.g.* division by the unknown quantity, in case this quantity was equal to nothing). If therefore we combine together two of these contrary propositions we must merely be prepared to interpret the combination as implying that there is no such thing as the subject of them:—whether or not there be anything corresponding to their predicate is not thus determined.

In ordinary Logic this difficulty seems to have evaded notice[1], doubtless owing to the fact that ordinary Logic seldom or never has to deal simultaneously with more than two propositions and the conclusion from them; these propositions being moreover of a very simple kind. Therefore any direct and glaring contradiction, such as that in question, could hardly present itself, for it would at once strike the attention, and the premises would have been reconsidered. But any extended System of Logic, which ventures to deal with combinations of several complex propositions, must make up its mind how to deal with this measure of contradiction and of what would commonly be regarded as incompatibility. Suppose, for instance, that we have three or four propositions, involving possibly half a dozen terms, no

[1] There is a suggestion in this direction:—as of what indeed in Logic is there not a suggestion?—by Leibnitz. After saying that from the premises '*a* is *b*', '*c* is *d*', we can deduce the conclusion that '*ac* is *bd*', he applies this to the case of *a* and *c* being incompatible: "Circulus est nullangulus. Quadratum est quadrangulum. Ergo circulus-quadratum est nullangulus-quadrangulum. Nam hæc propositio vera est ex hypothesi impossibili" (*Specimen demonstrandi*, Erdmann, p. 99). The only distinct notice that I have met of the subject discussed above is in an interesting and highly suggestive article by Mr C. S. Peirce in the *Journal of Spec. Phil.* for 1868.

untrained acuteness would be capable of detecting at a glance whether or not they involved amongst them this measure of contradiction. The example given on page 156 is a case in point. There are only three propositions in it, involving four terms; but who would undertake to say at once, on looking at it, what was the solution? We must always be thus prepared to find that the non-existence of one or more of the classes involved in the combination, as subject or predicate, is all that is required to harmonize them satisfactorily with one another. Suppose we put together several Acts of Parliament, Rubrics or Canons Ecclesiastical, or rules of some club, it might so happen that they turned out, on comparison, to be in this way irreconcilable with each other. No doubt if this were found to be the case in practice we should take it as a sign that the rules needed remodelling. But if we chose to adhere to them as certainly true, and to treat them as we should treat such statements if given us through a set of equations, it might well be that the irreconcilability would disappear at once on our abandoning some class which we were in no way pledged to retain. The confusion would have only arisen from the fact that we were trying to dispense with the universal and necessary, but tacit, proviso that the various classes with which we deal are only referred to conditionally, 'if they exist'; and that in the case in question some of them did not exist[1].

[1] Jevons maintained, as already remarked, that we are not to allow this extinction either of any simple class or of its contradictory. Thus of the example:

$$\begin{cases} a = b + c\bar{b} \\ b = \bar{c} + c\bar{d} \\ \bar{c}\bar{d} = 0 \\ ad = bcd \end{cases}$$

(the notation is mine) which demands $a, b, c, = 1$ to satisfy it, he says "we find that there cannot be any $\bar{a}$ at all without contradiction, whatever may be the meaning of this result. It means doubtless that the premises are contradictory" (*Pure Logic*, p. 64).—It means simply that 'everything is a' (and also b and c), a

The only sort of contradiction amongst our propositions which we really have cause to fear, that is, the only one which we must peremptorily decline to admit under any interpretation, is one of a much more thoroughgoing kind than the common contradiction of ordinary predication. It would be one which tried to empty and to occupy a compartment simultaneously; either directly (by combining a particular and a universal), as if we persisted that there is xy and that there is not xy; or indirectly, as if we set about denying the existence of all the four classes xy, $x\bar{y}$, $\bar{x}y$, and $\bar{x}\bar{y}$. It must not be replied here that in admitting this we are reintroducing the just rejected doctrine that such propositions as 'All X is Y' and 'No X is Y' are absolutely incompatible. The two cases are not in reality on the same footing. In the latter case we were able to make the reply:—By all means let us admit both the propositions simultaneously, provided they are offered to us under the tacit condition that there may not really happen to be any such thing as an X; a condition which in this case we should require to make use of. But when we are dealing with what I have called compartments rather than classes, that is with formal instead of material divisions, no such reply is available. The compartment is formal and necessary; we

perfectly admissible conclusion. To reject such a conclusion is to confound the formal possibilities of our notation with the material conditions of our data. Such a principle of rejection is difficult enough to adhere to within the range of ordinary propositions, but in the case of the extended propositions of the Symbolic Logic it would become simply suicidal. Excellent as much of Jevons' work is,—especially as regards the principles of physical and economical science,—I cannot but hold that in the domain of Logic his inconsistencies and contradictions are remarkable. The same man who has thus laid it down that no single term can be allowed to vanish has asserted "the important logical principle that all propositions ought, strictly speaking, to be interpreted hypothetically" (*Studies*, p. 190).

can never admit the assumption that it may possibly not exist. *Things* exist indeed (in the sense of which has been already fully explained): this amount of assumption we certainly do make. And these things must either possess any assigned attribute or not possess it; which is equivalently expressed by saying that they must certainly fall into one or other of these compartments. The analogous retort therefore here, in order to be effective, would have to be that there *are* no things whatever, at least within the universe in question.

The above statement,—that it is impossible for all our class compartments to be simultaneously empty,—was criticized with some pleasantry by Mr Bradley (*Princ. of Logic*, p. 153) and with some argument by Dr Keynes (*Formal Logic*, p. 145). The latter maintained that "with very limited universes this does not seem a legitimate assumption." His example may be stated thus. Let our universe be confined to those who pass a certain examination, and suppose that we receive the four items of information: that there are no Trinity-men in the first class, and none in any other class; and the same of the non-Trinity men. These results would yield (say) xy, $x\bar{y}$, $\bar{x}y$, $\bar{x}\bar{y}$, all $=0$. That is, *all* the compartments in our universe would be emptied out. This is so; but how would the problem have been stated? The natural statement would have been, Let z stand for those who pass, so that $z = xy + x\bar{y} + \bar{x}y + \bar{x}\bar{y}$. The successive destructions would then yield the result, $z = 0$, or $\bar{z} = 1$. In other words the problem should have been stated in *three* terms, and we should then have had nothing but the familiar result of one term, z, vanishing, and its contradictory, $\bar{z}$, filling the universe. In order to obtain Dr Keynes' result, we must start with the deliberate resolve that 'those who pass the examination' shall con-

stitute our universe, though we have no ground for assuming that there are any such. We should, so to say, be riding for a fall, by courting the result[1] that all the four elements on the right side of the equation, $1 = xy + x\bar{y} + \bar{x}y + \bar{x}\bar{y}$, shall $= 0$.

Such riding, however, may be quite justifiable when we wish to decide experimentally the nature of the ground on which we land. I think that this way of presenting the matter does raise some interesting questions. The first of these concerns our fundamental condition, $1 = xy + x\bar{y} + \bar{x}y + \bar{x}\bar{y}$, and involves one of those "limiting cases", which, if they are often merely curious, are sometimes very instructive. We have repeatedly said that by the universe we mean *all*, whatever the "all" may be with which we happen to be concerned. Can this ever, in the limit, shrink up to *nothing?* It seems to me that though we may approach this limit as closely as we please, yet when it has been actually attained we should find it scarcely possible to extract any significance out of the assertions thus involved. Amongst other things we should thus have to admit that $1 = 0$. What would be the physical or mathematical counterpart to this? What we generally denote by 1 is a *unit* of some kind or other, but of a magnitude not necessarily determined for us. Conceive now that our unit becomes continually smaller, and at last in the limit vanishes. All multiples of it, and therefore all magnitudes assigned in multiples of it, would also vanish; and we should be led to the conclusion, as an applied result, that $1000 = 1$. The mathematician would of course reply that the significance of a 'limit' is to be extracted as we approach it, not when

[1] Dr Voigt (*Algebra der Logik*) has prominently and very justly insisted that the possible reduction of a system of equations to the result $1 = 0$, is the one criterion of inconsistency in the Symbolic Logic.

we have actually got there; and that the attempt to go direct to the extreme case and interpret it as it stands, will probably lead to absurdity.

This brings us to a further consideration. If all four combinations of x and y might possibly be equated to zero, the destruction of three of them would not necessarily demand the saving of the fourth. That is, we could not deduce a particular proposition (in our interpretation of it) from three universals. This was the connection in which Dr Keynes raised the question, and he consistently drew the conclusion thus stated. It seems to me, however, that no mere *diminution* of the universe would justify this inference: it could only be justified, if at all, when that universe has absolutely vanished. And at this point the significance of the particular proposition itself has also vanished. It is as if, in our diagram, we drew a circle outside the x and y figures, in order to indicate our universe; and then supposed this to shrivel up until it, and the included circles, vanished in a mathematical point. So long as anything was left the particular proposition would cling to its meaning; but when all had disappeared it would be impossible intelligibly to assert that the absence of anything from three of the compartments necessitated its presence in the fourth.

We must now turn to another somewhat similar topic in which considerations of a like nature present themselves. On the same grounds on which we have to demand a discussion and revision of the meaning and consequences of contrariety and contradiction in propositions, when they are found in combination, we must also demand a reconsideration of what is to be understood by the 'dependence' and 'independence' of propositions. Here, as in the last case, the question is really presented to us in ordinary Logic and not in Symbolic only, though it does not become prominent

within the range of the former. The forms of dependence indeed with which the common system mostly has to deal are simple enough. 'All A is B', and 'Some A is B'; 'No A is B' and 'Some A is not B' are obviously dependent, in the sense[1] that the second member in each pair is included in, or inferrible from, the first. In the same way 'X is neither Y nor Z' includes within it 'X is not Y'.

But if we admit that hypothetical interpretation of all Universal propositions, which is under discussion in this chapter, we shall readily see that once started on that track we cannot stop so soon as this. Just as 'inconsistency' had to contract its boundaries within the limits of what was in a sense rigidly formal, so will 'dependence' have to extend its boundaries up to the same limits. Here too the mathematician, and the mathematically minded, will find nothing perplexing. All that is required is that we should adhere rigidly to the assigned meaning of our symbols, and not suffer ourselves to give way to any of those tacit implications which are of course inevitable in ordinary speech. Our x and y are undoubtedly symbols standing for classes, but they do not commit themselves to saying that such classes are to be found.

Take the extremest case possible of apparent independence between negative propositions, for instance, 'No A is B', 'No C is D'. Are even these really independent of each other, in the sense of not possessing any common implications? Though neither of these two propositions can be certainly shown to cover any part of the ground occupied by the other, yet they *may* prove to do so for anything that we can tell to the contrary. Hence there is one common conclusion that can equally be deduced from either of them, and accordingly, to that extent, we must not speak of them

[1] That is, in the ordinary logical sense, not the symbolic.

as independent. This common conclusion is of course, 'No AC is BD'. For if no A whatever is B, this must hold of the A that is C, viz. of AC. Similarly, if this AC is no D whatever, it cannot be the D which is B, viz. BD. We may infer therefore (expressing ourselves fully and cautiously) 'No AC (if such there be) is BD (if such there be)'. But this is the interpretation, as we have seen, under which it was agreed that such a simple and categorical negative as 'No X is Y' was to be accepted.

We need hardly remark that no logician will have any occasion to feel himself troubled by material difficulties in any of these cases. If from 'No men are green', 'No horses are red', we should shrink from drawing the conclusion 'No men-horses are green-red', on the ground of its absurdity; we must remember that from 'No Americans are monarchists', 'No capitalists are unselfish', the less unreasonable conclusion that 'No American capitalists are unselfish monarchists', would only stand upon the same footing of provisional acceptance. Both alike are unassailable in so far as they deny, and on our interpretation they are too cautious to do anything but deny.

All this becomes clear and straightforward when regarded in the light of our symbols. 'No A is B' simply blots out all the occupants of the compartment AB, as 'No C is D' blots out all those of CD. But AB, as a compartment, must contain the four divisions made by C and D, any of which *may* be occupied; just so CD contains those made by A and B, which may equally be occupied and have accordingly to be reckoned with. The subdivision $ABCD$ being therefore contained within both AB and CD, it is clear that the original propositions are to that extent not independent of one another. If we were determined to make the second proposition certainly independent of

the first we should have to limit its range by the insertion of another clause. We must phrase it '*No C which is not AB is D*'. When they are so phrased each proposition keeps itself entirely clear of the ground occupied by the other, and nothing is denied twice over. Of course in such a simple example as this it is easy to see by inspection what is the common part, in other words, what is the extent of the dependence between the two propositions. But when we have to deal with groups of propositions and considerable numbers of terms, no acuteness would suffice without the aid of systematic rules. It was one of the characteristics of Boole's system that it furnished such a rule, so that by a simple and regular procedure every fragment of surplusage involved in the whole group of propositions could at once be detected and brought to light. It was the fulfilment of the requirements for avoiding such surplusage of statement as this that Boole contemplated when he discussed some of the conditions of a perfect language. What was supposed to be demanded was a system of statements which should be complete, and as far as possible symmetrical; which should leave no gaps between any of them unaccounted for (except of course where we are really in ignorance) and which should never interfere with each other by going twice over the same ground.

The reasons for adopting this interpretation of the import of propositions will only be fully appreciated after the examination of a number of examples, but enough has already been said to afford a fair justification. It is not for a moment maintained that this view is entirely in accordance with popular impressions; though if the choice lay between it and that which seems consistently deducible from the rules of ordinary Logic, the former would, I think, be found to have the majority of popular votes. For if the

ordinary person, after hearing that 'All X is Y', would be disturbed by the admission that perhaps there were no X's, he would probably be still more disturbed if required to admit that there were certainly things which were not X and not Y, as well as X and Y. Moreover, as it happens, we can readily rectify even this divergence from popular association. We only say that 'All X is Y' shall, in the absence of express assurance, be taken as equivalent to 'There is no X which is not-Y'. If such assurance be given, it is to be regarded as an independent assertion and must be dealt with as such. Of course this would complicate matters, and as it hampers the freedom and generality of our rules we shall make a practice of avoiding it. But examples can readily be given, and some will be given, to show how the admission of any such intimation into our data would lead to modification of our inferences.

CHAPTER VII.

SYMBOLIC EXPRESSION OF ORDINARY PROPOSITIONS.

HAVING thus cleared the ground in respect of the general existential import of our propositions, we are in a position to complete the discussion as to the best mode of expressing symbolically the familiar propositions of ordinary Logic.

I. The Universal Affirmative, 'All x is y'. We have seen that we are involved in numerous perplexities if we permit this form of proposition to imply the existence of either x or y. It can therefore best be understood in the conditional sense that if there be x's and y's then all the x's must be y. How are we to put this meaning into symbols? So long as we keep to the equational rendering three equivalent forms may be suggested, one of them negative and the others positive.

(1) The negative expression is the one referred to repeatedly in the course of the last chapter. We simply frame it, $x\bar{y}=0$; viz. 'No x is not-y'. This seems to cover unambiguously all the meaning which we want it to cover, and nothing besides. It is equally true whether x is the whole of y or only a part of y. Again if there be no y, then not-y being 'everything' there can of course be no x;

but if there be y there need not consequently be any x. Also if y be 'everything', then the x, if there be any (which is not necessary), must be y. This simple negative form therefore carries naturally with it all the signification which, as we have seen, should be read into the Universal Affirmative, and no more than that.

(2) The form adopted by Boole is $x = vy$, where v is to be regarded as an indefinite class symbol. At least this is the form with which he starts in order to express such propositions, the form which he deduces as a conclusion from his rules of operation being the unfamiliar one, $x = \frac{0}{0} y$. The nature and justification of the rules by which this latter mathematical form is deduced will occupy our attention hereafter, but the meaning of the symbol can be easily assigned. In mathematics $\frac{0}{0}$ represents the *absolutely indefinite* in respect of numerical magnitude, standing for anything between 0 and infinity. We might propose to take it then in Logic in the proper analogous sense, with the one necessary restriction that our universe ranging (symbolically) only from 0 to 1, instead of from 0 to ∞, we must keep within those narrower limits. In that case $\frac{0}{0}$ would be absolutely indefinite between 0 and 1, that is, it would denote a class which may be anything between '*nothing*' and '*all*', inclusively. But although if we thus borrowed the symbol at once as a suitable sign of the indefinite, we might offer a fair defence of our choice, it must be insisted on that we are in no way whatever dependent upon a second-hand interpretation here. The symbol has an undeniable logical signification of its own. Just as $\frac{x}{y}$ denotes the class which on restriction by y will reduce to x, so does $\frac{0}{0}$ denote the class which on restriction by 'nothing', that is by our taking none of it, will reduce to nothing. But clearly any class whatever

will do this, so that $\frac{0}{0}$ stands for any logical class whatever. That is, it is perfectly indefinite.

The reader must not however mistranslate this symbol by the introduction of quantitative considerations. In mathematics it signifies indefiniteness in respect of continuous magnitude: in logic it signifies indefiniteness in respect of the number of class compartments involved, and therefore only comprises a limited number of elements in a discontinuous magnitude. Prefix the symbol $\frac{0}{0}$ to a solitary class term x, and the uncertainty extends merely to rejecting or not rejecting it: introduce a second term y, and there are two subdivisions to which such uncertainty applies. If there are four terms introduced, x will have eight subdivisions, and the indefiniteness of the symbol becomes of a more distinctly numerical kind.

It must be clearly understood therefore that if we put v as an equivalent for $\frac{0}{0}$, this v is by no means a substitute for the 'some' of ordinary logic, since it includes 'nothing'; still less for the 'some' of ordinary language, since it also includes 'all'. It is the more necessary to call attention to this since Boole himself has repeatedly treated this v as equivalent to 'some', which I can only regard as an oversight. There is, in fact, no exact equivalent in our language for this form $\frac{0}{0}$, which seems to me an argument in favour of its introduction; for its very unfamiliarity calls attention to the various implications which we are to admit into, or exclude from, the meaning of the proposition better than a mere arbitrary letter like v can do. But the prejudices which it is likely to excite are so strong that I must confess to not having always had the courage to press its just claims to employment.

Before passing on to discuss an alternative form a difficulty must be signalled here. We say that $x = \frac{0}{0}y$ (this $\frac{0}{0}$ being absolutely indefinite within our limits) represents the

Universal Affirmative, and that only. Now $\frac{0}{0}$ includes 0. Hence in this limiting case the proposition would stand $x = 0y$. The inexperienced reader might translate this into 'All x is *no* y', and thence take it as the equivalent of 'No x is y'. In other words he might conclude that this way of symbolizing the Universal Affirmative had broken down, since it was found to include the Universal Negative also. This would be a mistake, involving the same play on the word '*no*' as the familiar problem, which it is said took Whately's fancy, about 'no cat' having more legs than 'a cat'. The equation $x = \frac{0}{0}y$ identifies the whole of x with an uncertain portion of y; when $\frac{0}{0} = 0$ it identifies this whole with a part of y which is *nothing*, in which case of course x itself $= 0$. 'All x is no y', if we are to use the phrase in this case, refers the x still to 'a part' of y, but *declares that part to have vanished*; it does not refer it to any part of not-y, and this last would be necessary in order to identify it with the Universal Negative[1].

(3) There is a third form, viz. $x = xy$, which was probably first employed by Leibnitz[2]. It is discussed by Ploucquet[3], and is occasionally made use of by Lambert[4].

[1] It was just at this point that an ingenious scheme by Holland (Lambert's correspondent: see on, ch. xx) went wrong. He proposed $\frac{S}{p} = \frac{P}{\pi}$ as a general propositional form, in which p and π are to lie between 1 and ∞. This of course is the equivalent of $vS = vP$ when v lies between 1 and 0, as on Boole's notation. He then examines the nine cases yielded by putting p and π respectively equal to unity, greater than unity, and equal to infinity. But when he gets $S = \frac{P}{\infty}$ or $S = 0P$, he does not translate this as he should, 'All S is the same as *no* P', *i.e.* is *nothing;* but simply 'All S are not P', *i.e.* 'No S are P'. Similarly $0S = 0P$ is translated, not as absolutely indefinite, but as 'All not-S are all not-P'. This is to confound $0S = 0P$ with $\bar{S} = \bar{P}$.

[2] *Difficultates quædam logicæ*, 'Omne A est B; *i.e.* æquivalent AB et A'.

[3] *Sammlung, &c.* p. 262.

[4] *Sammlung*, p. 212; *Log. Ab.* I. 23.

In no effective way does it differ from the form last mentioned; and though not primarily employed by Boole as representative of the proposition, is constantly presenting itself in the course of his analytical processes.

This form is best known at present from its systematic employment by Jevons, who did not seem to be aware that it had been so much employed before him. In supporting it against Boole's $x = vy$, he objects to this latter on the odd ground of its indefiniteness;—indefiniteness, that is, in respect of the extension of the predicate; this characteristic being the very thing which it is bound to possess if it is to coincide in signification with the ordinary proposition[1]. He declares that he will "throughout this system of Logic dispense with such indefinite expressions". If it were really the case that we thus introduce greater definiteness of statement it would seem to be a conclusive objection against the form in question, instead of an argument in its favour; for the proposition would thus be suffered to express more than it has any right to express. A little consideration will however show that the two forms are in every way exactly equivalent. For, starting with either $x = vy$ or $x = \frac{0}{0}y$, multiply each side by the factor $\bar{y}$, and we have $x\bar{y} = 0$: that

[1] *Principles of Science*, p. 41. Others also have failed to see the substantial identity of the two forms. Thus A. Riehl, in a review of the *Principles of Science*, says, 'Boole employed the indefinite symbol v for denoting a partial identity; whilst Jevons makes the definite signification of such an identity visible by his notation' (*Vierteljahrsschrift für wiss. Phil.* 1876). The same idea is repeated further on in words which, considering what they assign to Hamilton (it is a pity he was not alive to read it), I quote verbatim: "Hamilton und Boole schreiben diese Urtheile in der Form $X = vY$, allein diese Bezeichnung lässt unbestimmt welcher Theil von Y, X sei". The supposition, that we can in this way secure greater definiteness of statement, is really nothing but an old joke expressed in general symbols:—'What functions does an archdeacon perform? archidiaconal functions':—What part of y does x cover? the x-part.

is $x(1-y)=0$, or $x=xy$. On the other hand if we start with $x=xy$, and eliminate x (by a process to be hereafter explained) we are led at once to $x=\frac{0}{0}y$ or $x=vy$.

These three forms then are exactly equivalent and convertible. Which therefore, it may be asked, is the best for our purpose? So far as there is any difference, the distinction lies between (1) on the one hand, and (2) and (3) on the other; though this is merely the distinction between the negative and its corresponding positive form. As between these we must make our choice according to circumstances, sometimes one and sometimes the other happening to be the most convenient in the statement of a problem. As between (2) and (3) the preference should in strictness be given to (2), at any rate as the primary form of statement. It is to be preferred on the ground that it more prominently calls attention to the indefiniteness in respect of the distribution of the predicate which, as we have seen, the form $x=xy$ somewhat tends to conceal. As a primary expression the form $x=xy$, for 'All x is y', should be slightly distasteful alike to the logician and the mathematician; for the former shrinks as much from introducing the term to be defined into a definition, as the latter does from offering an implicit equation in place of an explicit when this latter is available.

It should be observed that, though these two latter forms are thus exactly equivalent, it may still be the case that in the process of working there are reasons for preferring one or other of them. Thus, in dealing with the simpler kinds of statement, we shall often find that $x=xy$ is an easier form to deal with. But it cannot be very readily used unless the term on the left is single. Thus $x\bar{y}+y+z\bar{x}\bar{y}=\frac{0}{0}W$, if we wish to state it as a single proposition, would have to be written out in the form $x\bar{y}+y+z\bar{x}\bar{y}=(x\bar{y}+y+z\bar{x}\bar{y})W$; nor would the amount of symbols demanded for its ex-

pression be economized by breaking it up into three distinct propositions.

II. The Universal Negative: 'No x is y'. About this there is less difficulty. When I say that 'No x is y', it will be admitted, I apprehend, that it ought strictly to be a matter of indifference to me whether x and y exist or not. If either or both of them be wanting, the proposition is certainly rendered none the less significant; for it is only the combination of the two that I am concerned to deny. If I were told to kill a dog I should certainly have a difficulty in executing the order in case no dog were found alive; but if I were told to make certain that no dog was in the garden, my task should surely be simplified on my finding that there was neither dog nor garden in existence, at the time and place in question.

Accordingly there would seem to be little opening for difference of opinion that the way to express such a proposition should be $xy=0$. Boole preferred to write it, at least as an initial step, in the form 'All x is not-y', $x = v(1 - y)$. He was doubtless influenced by feelings of logical conservatism: since such a form preserves at least the semblance of a subject and a predicate, instead of combining them into one whole and denying the existence of this combination.

III. So much for Universal propositions, the treatment of which is comparatively easy. We now come to particular propositions, and at once ground on serious difficulties; in fact it would not be too much to say that their adequate representation has proved a vexation to most thoughtful symbolists[1]. Part of the difficulty is one of mere ambiguity and affects the language of logic and of common life alike.

[1] I have purposely let the above phrase stand as it was originally written, but there is now practically universal agreement in the general view above explained.

What does 'some' mean? what cases does it cover? This depends upon whether we take it as 'some, it may be all' or 'some only'. With the former explanation 'some x is y covers the four following cases:—'All x is all y', 'All x is some (only) y', 'Some (only) x is all y', 'Some (only) x is some (only) y':—in fact it only excludes the case 'No x is y', of which it is the simple contradictory. With the latter explanation it would include only the last two of these four cases. The former is the ordinary logical explanation, and shall be adopted here.

When this obstacle is surmounted we come upon certain symbolic difficulties of representation of a more serious kind. Does the Particular Proposition give us any assurance of the existence of its subject and predicate? It seems clear to me that we must in this respect put it upon a different footing from the Universal Affirmative, and say that it does give such an assurance: on the ground that if it did not do so it would have absolutely nothing certain to tell us. Whatever shade of doubt may hang over the existence of x and y in 'All x is y', one result at any rate is certain, viz. that we thus extinguish $x\bar{y}$; and from this we may deduce the hypothetical affirmative form. But when the proposition 'Some x is y' comes to be affected in the same way, by a similar interpretation, it is paralyzed at once. It can extinguish no class and establish no class, and has therefore no categorical information to give. But that this characteristic of really establishing something must be admitted, and indeed made prominent in such propositions, was, I hope, fully made out in the last chapter. And this conclusion is supported by popular usage, for such propositions have, so to say, a tone of reality and of sober fact about them which cannot always be claimed for universals.

Boole's formula is $vx = vy$, where v is to be regarded as

an indeterminate class-term. This is obnoxious to various objections. The main difficulty, to my thinking, is in the interpretation to be assigned to this term v. We may call it an 'indeterminate' symbol, as Boole repeatedly does, but we cannot afford to let it really be so, or the proposition, as we have just seen, will break down and have no message to give us. We must expressly stipulate therefore that v shall not equal *nothing*. If indeed we really wanted to represent entire indefiniteness it would be better not to use the letter v for the purpose. We have already made acquaintance with a symbol which aptly signifies such indefiniteness. So if we wish to see how $vx = vy$ would look, when v was quite unrestricted, we had best write it at once $\frac{0}{0}x = \frac{0}{0}y$, and make what we can of it[1].

My own conviction is that we shall not, in this way, make

[1] In his *Principles of Science* (p. 41) Jevons declares, as already remarked, that he shall dispense with such indefinite expressions as 'some'. ("This can readily be done by substituting one of the other terms. To express the proposition 'All A's are some B's' I shall not use the form $A = VB$, but $A = AB$.") Unfortunately this is not what can be done in the case of a really particular proposition, as is exemplified by Jevons himself a few pages on, in treating the proposition 'some nebulæ do not give continuous spectra'. He says at once, "treating the little word *some* as an indeterminate adjective of selection, to which we assign a symbol like any other adjective, let A = some, B = nebulæ, &c." (p. 85).

This, since it gives the form $AB = AC$, for 'some B is C', is merely Boole's $vx = vy$ over again; or rather it is that form with one exception which here assumes a certain importance. It was shown in the last chapter that Jevons adopted a sadly hampering restriction by declaring that no simple class-term was to be equated to either 0 or 1. As a general restriction on our symbolic procedure this would be suicidal,—in fact it cannot be adhered to,—but in this special case it comes in serviceably, or rather half of it does. By forbidding the value $A = 0$, in $AB = AC$, we save 'some' from being 'none'; and this is well. But by forbidding $A = 1$, we prevent 'some' from being 'all', either in subject or in predicate, and this is by no means what we intend to imply.

much of the symbolic treatment of particular propositions; the fact being that they are, in their common acceptation, too quantitative for us. What Symbolic Logic works upon by preference is a system of dichotomy, of x and not-x, y and not-y, and so forth. The sort of propositions therefore that suit us best are those which yield two alternatives only, such as individual propositions:—A is B, A is not-B, and so on. But the particular proposition, in its common acceptation, slips in between these two and says '*Some* of the A's are, or are not, B'. And this we cannot represent symbolically, by the mere employment of a common class-term. Of course if these 'some' were indicated by a genuine class-term we could express them at once; they would then be marked off by C or D, and would become the CA, or the DA, or whatever it might be. But it hardly needs pointing out that we have then quitted the particular and taken to the universal again, for the CA is by no means the vague 'Some A'. If by 'some men have curly hair' I mean that *black* men have, I had better substitute 'black' for 'some', and so make a universal of it. But this completely alters the kind of proposition. 'Black men' are the objects common to the classes 'black' and 'man':—in contrast with this let us try to put ourselves in the position of taking the common part of the class 'man' and the class 'some'. The word 'some' marks, and most appropriately marks, those cases in which we are wholly without any other selective class-term which we could substitute for it: *e.g.* 'some throws of a penny will give heads', where it is impossible to substitute any more definite term for 'some'. The device by which we evaded, or thought we evaded, the indefiniteness of the predicate in 'All A is some B', by saying that the A is AB, fails us entirely here. We cannot conceal our ignorance in the case of 'Some A is B', where both subject

and predicate being in the same predicament of uncertainty neither can aid the other[1].

Another illustration may be offered to show how alien is the ordinary sense of 'some' from such symbolic treatment as that alluded to above. We know that not-A always means the *rest*, after A has been excepted; so that A and not-A together make up 'all'. Accordingly, if 'some' be marked by a class-term like any other substantive or adjective, 'not-some' ought to mean 'the rest', or 'all except that some'. We shall have instances hereafter of this treatment of a really indeterminate class-term, when it will appear that $\frac{0}{0}$ and $1-\frac{0}{0}$ are precisely equivalent in their signification. But it would surely be doing unnecessary violence to common usage to insist upon using the word 'some' here, and so to maintain that 'not-some' also meant 'some' instead of, as by invariable usage, meaning '*none*'. Turn it how we will, language and common sense rebel against the attempt to put the word 'some' on the same symbolic footing as any other substantive or adjective, by assigning it an ordinary class symbol.

[1] There is another form which has been adopted for particular propositions by Professor Delbœuf and Mr J. J. Murphy. Thus the latter writes 'some x is y' in the form $x-q=y-p$ (*Relation of Logic to Language*. See also *Mind*, no. v). The meaning of this is, that, since x and y must have some part in common, if we deduct the 'x that is not-y' (call it q) from x, and the 'y that is not-x' (call it p) from y, the remainders will coincide. The objection to this form seems to me to lie mainly in the conditions by which it must be propped up, for we must insist that q shall be contained in x, and p in y: facts which the symbols themselves do not indicate. When these conditions are secured, this exception by 'subtraction' becomes identical with exception by 'multiplication'; that is, the formula may be written $x(1-q)=y(1-p)$ which is formally identical with Boole's $vx=vy$. And then finally we encounter here, as there, the question as to what are the limiting values to be admitted for the term employed in subtraction or multiplication.

My own view is that we shall best succeed with these propositions by taking account of quite another side of their character. To secure as much as possible of the current signification of the particular affirmative, and to express this clearly in symbols, we shall do best to write it in the form[1] $xy > 0$. By this expression we clearly exclude the value 0, or *none*:—whether we also exclude the value 1, or *all*, is not of so much importance, and will be briefly noticed presently.

What we thus lay the stress on is the *existential* character of such propositions. The expression $xy > 0$ is the simple denial of $xy = 0$; and may be read off in words as 'xy is

[1] It deserves notice that Boole adopted the form $xy = v$ in his earlier work (*Math. Analysis of Logic*, p. 21) but afterwards rejected it in favour of $vx = vy$. But in neither context can I find any discussion of the real difficulty which arises when we are called upon to decide the limits of indefiniteness to be assigned to v. It may be remarked that Leibnitz with his usual penetration, had observed that particular propositions could be thus expressed. He says (*Difficultates logicæ;* Erdm. p. 102) that he had formerly adopted the following scheme: 'Omne *A* est *B*...seu *A* non *B* est non-ens: Quoddam *A* non est *B*...seu *A* non *B* est ens: Nullum *A* est *B*, erit *AB* est non-ens: Quoddam *A* est *B*, erit *AB* est ens'. So far as it goes this exactly coincides with the arrangement adopted above. He also expressly distinguishes the universal from the particular in the words, "prior loquitur de possibilibus, posterior de actualibus":—the very distinction here proposed. But he seems to have rejected this view to some extent afterwards, owing to the consequent difficulties, about the *conversion* of propositions, which we have already noticed. But I find his discussion of the subject somewhat obscure.

In quite recent times substantially the same arrangement has been adopted by Professor F. Brentano (*Psychologie vom empirischen Standpunkte*, 1874), but he does not extend it beyond the four familiar propositions. He announces it as a novelty which is to spread dismay among orthodox logicians. Like some other symbolists he springs to the conclusion that the new mode of notation is to supersede altogether the traditional one; instead of being, as I should say, an alternative method, not necessarily hostile to the old one, but more suitable for the treatment of complicated problems and broad generalizations.

something', *i.e.* is not nothing; or, more colloquially, 'There is xy', or 'x and y are sometimes found together'. The last two of these forms are of quite familiar occurrence in the language of common life, and would, I think, be naturally accepted as equivalents of the logical particular affirmative. Similarly, when we want to express the particular negative we adopt the form $x\bar{y} > 0$. This would with equal readiness be thrown into the familiar colloquial statements, 'There are such things as x's which are not y', or 'x's which are not y do occur sometimes'[1].

We have thus noticed two forms of particular statement, one of these being our substitute for the ordinary affirmative and the other for the ordinary negative of the logical treatises. But it is obvious that when we lay aside logical convention in our arrangements, symmetry and consistency will call for two more such forms. To make our system complete in this department we must exhibit it thus:—

$$xy > 0,$$
$$x\bar{y} > 0,$$
$$\bar{x}y > 0,$$
$$\bar{x}\bar{y} > 0.$$

The first three of this list belong to forms already discussed and familiar,—the third being merely 'Some y is not x', against the 'Some x is not y' of the second. But the fourth is not a familiar form, at least in logical treatises. It would commonly be expressed as 'Some not-x is not y', or 'There

[1] As Mr W. E. Johnson points out, an *equation* naturally suggests the possibility of each member being $=0$: thus $A=B$ is satisfied by $A=B=0$. But an *inequation* implies that the greater term has extension greater than zero. Thus $A>B$ can only hold if $A>0$. That is, if we want to lay stress upon the existential aspect of the particular proposition we naturally adopt the inequality symbol.

are things which are neither x nor y', or 'x and y do not comprise everything'.

This form of expressing symbolically the particular proposition, though not new, cannot be considered familiar. That it does not quite coincide with the popular view must be admitted, though I doubt if the departure here is wider than in the case of the common Logic. But as regards this latter I can never feel very sure, not having been able to determine what the ordinary technical usage may be. If some logician will undertake to resolve the difficulties indicated in the last Chapter (pp. 153—161), laying down rules which shall be consistent with some view of the nature of general names and propositions acceptable at the present day, I feel sure that every thoughtful reader will be grateful to him[1].

As regards the relation of this scheme to the conventions of popular speech, the case seems to me as follows. In all universal propositions,—including of course those with complex subjects and predicates—there is one constant element present, and one only, viz. a denial. This we preserve, writing 'All A is B' and 'No A is B', in the forms $A\bar{B}=0$, $AB=0$. In addition to this there is an element which is not constantly present, viz. one of implication as to the existence of A and B. This we reject, or rather we say that when required it must be expressly claimed. Similarly, in all particular propositions there is a constant element, viz. the affirmation of existence. Accordingly we write 'Some A is B', and 'Some A is not B', in the forms $AB>0$, $A\bar{B}>0$. But along with this there are many implications, partly connected with the word 'some' as meaning 'not all', partly with other words which though excluded from Logic are abundant

[1] Dr Keynes' *Formal Logic* gives an adequate discussion of these questions.

in common speech, such as 'many', 'most', &c. All these convey information, not merely as to the existence of the AB and $A\bar{B}$, but as to the *amount* of it present; and all these implications we reject, as being too quantitative.

In saying this, one point must not be forgotten. Whilst fully admitting that we depart from customary usage in our interpretation both of the universal and of the particular, it must be pointed out that in respect of one *relation* between the two, and this the most prominent relation, we do not depart at all. This is the relation known as 'diametrical opposition'. It is, for instance, universally agreed that 'Some A is B' is the simple denial of 'No A is B', just as 'Some A is not B' is of 'All A is B'. Now this relation is just what we retain; for $xy > 0$ is simply the symbolic negation of $xy = 0$, and $x\bar{y} > 0$ of $x\bar{y} = 0$.

The only point at which we come into serious conflict with traditional doctrines is in refusing to admit the inferences of subalternation; that is to deduce the particular affirmative and negative from the corresponding universals. From 'All A is B', we cannot infer 'Some A is B'; for if the one merely destroys $A\bar{B}$, and the other merely saves AB, and if these two classes are entirely distinct (as of course they are) then the two propositions clearly do not come into contact with each other at any point. I have purposely emphasized this objection in order to bring out the characteristics of this mode of treatment, but the difficulty is not so formidable as it may seem. Popular thought, when cautioned, would surely agree with us even here. Many universal propositions, or what the logician would treat as such, might be proposed from which every one would feel that it was very hazardous to infer a particular. We do not infer 'Some A is B' merely from 'All A is B', but from this proposition taken with its rider, 'and there *is* A, and B';

the inference therefore falls to the ground as soon as we agree to exclude this rider from forming any necessary part of, or appendage to, the proposition.

Whenever the existence of the subject and predicate is claimed and duly expressed, the inference follows on the symbolic notation as readily as on any other. 'All A is B, and there is A' would be expressed: $A\bar{B}=0$, $AB>0$; and 'Some A is B' would merely be the latter part of this expression repeated again, viz. $AB>0$. In other words the function of the particular affirmative on this principle is that of supplying and distinctly formulating an implication which common thought often makes as a matter of course[1].

This must suffice for the present treatment of these

[1] Herbart's view about the mode in which existential propositions are connected with categoricals is curious, and, though not the same as that described above, comes in contact with it at several points. He considers (*Einleitung*, § 63) that the scope of the predicate in affirmation is limited and conditioned by that of the subject. Therefore the greater the depth and the less the breadth of the subject notion, the less will be the breadth of the predicate. Now conceive the limiting case in which the subject *disappears*, so that we have in its place merely 'it is', or 'there is', connected with a predicate. We cannot suppose the predicate to have vanished too, or there would be no proposition (it would be the baffling result $0=0y$, as the limiting value of $x=xy$). On the contrary, the predicate being then unrestricted and unconditioned, stands by itself, and the sentence takes the form of an existential proposition. 'Jene verwandelt sich in das Zeichen von diesem wenn für ein Prädikat das Subjekt fehlt, und es entsteht auf die Weise ein Existentialsatz' (p. 105). Elsewhere (*Hauptpunkte der Logik*) he compares the condition of the predicate when the subject vanishes with that of the second member of the equation $S=x^a$, when $a=0$, and we have $S=1$.

I cannot accept this account of the process, and the whole discussion is as usual too much cast into 'Begriff' language to fit in readily with such a purely class theory as ours. But the passage deserves notice as one of the extremely rare instances, till recently, of the examination in Logic of one of those limiting cases with which every mathematician is so familiar.

propositions. Most symbolists, I think, however satisfactory they may consider their own solution of the difficulty to be, have been practically agreed in having comparatively little to do with them in the course of their work. And quite rightly so. Indeed I almost question whether, if the Symbolic Logic had been developed before the Aristotelian had acquired so firm a hold upon us, such propositions would have been admitted at all. To exclude them from our rules would only be a slightly greater encroachment upon the full freedom of popular speech than has been already brought about by the exclusion of such terms as 'many', 'most', and others of a somewhat quantitative character. Particular propositions, in their common acceptation, are of a somewhat temporary and unscientific character. Science seeks for the universal, and will not be fully satisfied until it has attained it. Indefiniteness indeed in respect of the predicate cannot, or need not, always be avoided; but the indefiniteness of the subject, which is the essential characteristic of the particular proposition, mostly can and should be avoided. For we can very often succeed at last in determining the 'some'; so that instead of saying vaguely that 'Some A is B', we can put it more accurately by stating that 'The A which is C is B', when of course the proposition instantly becomes universal[1]. Propositions which resist such treat-

[1] If we could *always* do this, particular propositions might of course be suppressed altogether; and this is possibly what Jevons meant in the passages criticized on page 181. But neither usage nor scientific requirements permit such an assumption. 'Some B' does not necessarily mean 'the B which is marked out by the possession of an assignable attribute A'. There may be no such common attribute whatever so far as we know (as in the instance of the throws of the penny offered above), except that which is mentioned in the predicate, in which case the conversion of it into a universal would merely result in an identical proposition. (The additional difficulties introduced by denying the values 0 and 1 to any simple class term are extraneous to the subject.)

ment and remain incurably particular are comparatively rare: *their* hope and aim is to be treated statistically, and so to be admitted into the theory of Probability. The relative importance of really particular propositions is, I think, much exaggerated in the common syllogistic treatment, where nearly half the members are particular. But this is almost unavoidable owing to the extreme narrowness of that scheme. We cannot afford to be very scrupulous in what we reject when we are confined to only three terms and propositions, and are not allowed to resort to negative subjects and predicates[1].

The following table will serve to display the principal results obtainable on the lines above laid down. We are supposed to have two terms before us, x and y; and to be considering only the forms assumed by a simple compound of these or their contradictories. We may equate any one of these forms to either 0 or 1; or we may deny that they are to be so equated. The result will be as follows:—

$xy=0$,	$xy>0$,	$xy=1$,	$xy<1$,
$x\bar{y}=0$,	$x\bar{y}>0$,	$x\bar{y}=1$,	$x\bar{y}<1$,
$\bar{x}y=0$,	$\bar{x}y>0$,	$\bar{x}y=1$,	$\bar{x}y<1$,
$\bar{x}\bar{y}=0$,	$\bar{x}\bar{y}>0$,	$\bar{x}\bar{y}=1$,	$\bar{x}\bar{y}<1$.

The significance of these various forms will be quite plain from what has been already said. They may be regarded as elementary statements, and as containing all the elementary

[1] The alternative which every symbolist has to face here may be concisely expressed thus:—If he admits the values 0 and 1 as possible to every class term,—or if, whatever his notation may be, he does the equivalent of this,—how does he express the conventional sense of particular propositions? If he denies these values how does he express *any* proposition of a complex kind? I accept the former alternative, and meet the difficulty by admitting a third expression for propositions which are really particular, viz. $x>0$.

statements obtainable with a single compound of two class-terms.

(1) In the first column we declare that such and such a compartment is empty, that is, that the corresponding class is unrepresented in our universe. This grouping of propositions of course differs considerably from the customary schedules. The first of the four is always naturally couched in a negative form in ordinary language and logic, whereas the second and third generally appear as affirmatives. The fourth again has no precise logical equivalent, the popular equivalent being 'There is nothing but what is either x or y'. I call attention the more expressly to this point as it enforces the opinion, laid down in the introductory chapter, that there can be no absolute or universally appropriate arrangement of propositional forms. The number and grouping of our forms must depend upon the particular view we take as to what should be the import of a proposition.

(2) In the second column we declare that such and such a compartment *is* occupied, that is, that the class is really represented. The only trouble here is in settling the degree of indefiniteness to be assigned to the expression > 0. Fortunately for us, in such a notation as this, there are almost no acquired associations to be attended to, so we may define freely according to our judgment. That being so, it would seem best to lay it down that the expression shall be perfectly indefinite, except in so far as it excludes 0. The exact meaning of this form of proposition is that a portion, at any rate, of the things in our universe are found to belong to the class in question; the remainder of them being distributed in some way, we know not how, amongst the other three conceivable classes. The particular propositions of ordinary logic are best assigned to this class; for, though

not always precise in their signification, yet when they are made precise they most naturally take up the meaning here assigned.

(3) In the third column we declare that the compartment in question is not only occupied, but exclusively occupied: that a given class, and this class only, is represented in our universe. This is therefore an extreme or limiting case of the second column. We make a wide departure here from the familiar forms of the ordinary logic, for no one of its recognized propositions coincides with anything in this column. Popular language would express $xy=1$ in the words 'Everything is both x and y'; whilst $\bar{x}\bar{y}=1$ would yield 'Everything is neither x nor y', or 'Nothing is x and nothing is y'. It may be remarked that we called these forms 'elementary' a page or two back. All that was meant by this is that they may be considered elementary in so far as they deal with a single compound only, such as xy. Our table contains all the simple equations and inequations which can be constructed with these as elements, taken singly.

(4) The fourth column stands to the third in the same relation that the second occupies towards the first. In $xy=1$ we declare that xy is 'all', that there is nothing besides it: in $xy<1$ we declare that xy is *not* all, that there are other things besides it. The one simply contradicts the other. Naturally the forms of common speech do not supply any very simple equivalents here. Thus, $\bar{x}\bar{y}=1$ asserting that 'Everything is neither x nor y', $\bar{x}\bar{y}<1$ asserts that, 'Not-x . not-y does not exhaust the totality of things'.

It must be observed that every one of the above forms gives rise to a corresponding alternative. The first and third of the above columns are thus connected, as are also the second and fourth. This result follows symbolically

from the fundamental formula connecting the totality of class-terms, $xy+x\bar{y}+\bar{x}y+\bar{x}\bar{y}=1$; or, logically, from the fact that every existing thing must belong to one or other of the four classes thus indicated. In saying this it is not meant that any one of the four equations in the first column is the actual alternative or equivalent of any one of the second; but that we can find one of the same general form. The equation of a single term to 0 is equivalent to the equation of a group of three terms to 1, and so on.

There is no need to go in detail through the applications of this principle[1], which the reader will easily work out for himself; but one or two cases may be taken as examples. For instance, $xy=0$ yields the alternative $x\bar{y}+\bar{x}y+\bar{x}\bar{y}=1$, these two forms being precisely equivalent and convertible. Put into words this amounts to saying that it is exactly the same thing to assert that 'No x is y', or that 'Everything is either x and not y, or y and not x, or neither x nor y'. Precisely similar is the result of starting with one of the elementary forms in the third column. We get, as before, three alternatives in the equivalent result; with this difference however, that, negative disjunction being distributive, *all* these alternatives must be equated to 0. That is, if $xy=1$, then $x\bar{y}+\bar{x}y+\bar{x}\bar{y}=0$; a conclusion which, as will be shown hereafter, necessitates the three separate inferences, $x\bar{y}=0$, $\bar{x}y=0$, $\bar{x}\bar{y}=0$. Logically, this is clear enough; for if everything is xy, then certainly nothing is either $x\bar{y}$, $\bar{x}y$,

[1] These considerations form the basis of the very ingenious symbolic method of Dr Mitchell (see Johns Hopkins *Studies*). Instead of starting, as is generally done, with the series of negations, he starts with the complement of the remaining terms. All these terms are placed on one side of the expression, which is then equated either to (say) 1, as representative of the universe, when dealing with universal propositions; or to (say) u, between 0 and 1, when dealing with particulars. A few words of description of this scheme are given in the concluding chapter.

or $\bar{x}\bar{y}$. To crowd everything into one compartment is to empty out all the others of the set.

There is a close analogy between the relations of the terms in the second and fourth columns. From $xy > 0$ we may deduce the equivalent $x\bar{y} + \bar{x}y + \bar{x}\bar{y} < 1$. The meaning of this is readily assigned, for just as $xy > 0$ asserts that there *are* such things as xy, so the longer alternative asserts that there is something else than either $x\bar{y}$, $\bar{x}y$, or $\bar{x}\bar{y}$. It assures us that these three alternatives do not exhaust the universe. Similarly if we start with one of the inequations of the fourth column; for $xy < 1$ yields as its equivalent $x\bar{y} + \bar{x}y + \bar{x}\bar{y} > 0$. That is, if xy is not all, then there must exist a representative of some one or more of these three alternatives. The reader will carefully distinguish between this case of true alternation and the case of negative disjunction discussed just above. The establishment of *any one* of the three terms involved will serve to justify the expression $x\bar{y} + \bar{x}y + \bar{x}\bar{y} > 0$.

This will be a convenient place for noticing the opinion which has been repeatedly expressed about Boole's system,—and presumably about most other analogous systems of Symbolic Logic,—that we are thus forced to adopt the Hamiltonian doctrine of the Quantification of the Predicate. Thus Mr Lindsay says, "The doctrines contained in this New Analytic of logical forms lead directly to the theories of Boole and Jevons. A leading characteristic of the doctrine of the Quantification of the Predicate, and other recent theories of a similar kind, is the attempt to assimilate all propositions to the type of mathematical identities...[1]"; and Jevons goes further by declaring that "Dr Boole, em-

[1] *Translation of Ueberweg's Logic*, p. 568:—I am not sure to what extent Mr Lindsay is responsible for this part, as it is actually contributed by another writer.

ploying this fundamental idea [of Quantification] as his starting-point", arrived at such and such results[1].

The assertion that Boole's system is in any way founded on the doctrine of the Quantification of the Predicate,—is, in fact, not directly hostile to that doctrine,—is so surprising that one is inclined to suspect some confusion of meaning. I may therefore remark that what I understand by the doctrine is this:—Whereas the ordinary forms of proposition leave it uncertain whether we are speaking of the whole predicate, or part only, in affirmation, and decide that we must be speaking of the whole predicate in negation; we thus leave four possibilities unrecognized: that in fact we *must* think the predicate either as a whole or as a part, in both affirmation and negation alike. Moreover, since what exists in thought should be expressed in words, a really complete scheme of propositions demands, and is satisfied by, eight forms. There is surely no doubt that this is the sense in which Hamilton, and his authorized exponent Baynes, understood the doctrine.

[1] *Subst. of Similars*, p. 4:—Boole himself expressly stated that he took the four old forms of proposition "with little variation from the Treatises of Aldrich and Whately". (*Math. Anal. of Logic*, p. 20.) In his discussion of the Syllogism, in the *Laws of Thought*, he does, it is true, give an eight-fold scheme; but this is the scheme not of Hamilton but of De Morgan. The extra forms of this scheme are obtained, not by quantifying predicates, but by introducing negative terms. Boole takes the trouble to point out that this is *not* the scheme of Hamilton. The subject is of no particular importance, except in the interests of clear thinking, but as the confusion is perpetuated to this day I may just point out that it is not merely inconvenient, but actually *impossible*, to express in strict Boolian notation any of the distinctive Hamiltonian forms: *e.g.* All *X* is some (only) *Y*. Of course, if by Quantification of the Predicate is meant some doctrine different from that which Hamilton,—the undoubted inventor of the phrase,—intended, and taught by it, I have nothing more to say.

Now though it seems hard upon ordinary predicates thus to charge them with secretly quantifying, it may be urged that at least they have nowhere denied that they do so. But with Boole's system it is otherwise. If the wit of man had sought about for some expression which should unequivocally and even ostentatiously reject this doctrine, what better could be found than $x = \frac{0}{0}y$ for such a purpose? So far from quantifying the predicate, by specifying whether we take *some* only or *all* of it, we select a form which startles the ordinary logician by the uncustomary language in which it announces that it does not mean to state whether some only, or all, or even *none* is to be taken. The negative equivalent, $x\bar{y} = 0$, is just as resolute not to commit itself on this point; whilst, as was pointed out, $x = xy$ is a precisely synonymous expression. It is difficult to conjecture how these symbolic forms could be thus identified with Hamilton's doctrine, unless by a hasty conclusion from the fact that both systems adopt the equational form.

I objected strongly, in the first edition, to the attempt of Boole and Jevons to grapple with particular propositions by the use of ordinary class terms and a merely two-fold system of predication. But I then adopted the expression $xy = v$, for 'Some x is y'; the essential characteristic of v being that it should not be equated to 0. But this notation is awkward. The essence of the particular proposition consisting in this assertion of existence it is best to make this characteristic prominent, which is better effected by the formula $xy > 0$.

Equivalent expressions are employed by other symbolic logicians. Thus Mr Peirce extends the plan of writing a bar over a term (as an indication of negation) to the case of the copula. With him 'All x is y' is written, $x \prec y$: put a bar over this ($x \overline{\prec} y$) and we have 'All x is *not* y',

viz. 'Some x is not y'. Professor Schröder adopts an equivalent expression, modified in accordance with his own notation. This of course meets the case of the two ordinary particular propositions, since they are respectively the negations of the two universals. Such a plan is especially convenient when we simply want to deny the identity of two class terms, and I shall occasionally adopt it for this purpose.

Thus $x=y$ identifies the classes x and y, and is therefore equivalent to the two propositions, 'x is y' and 'y is x'. Write $x \neq y$, and we express the denial of this identity, *i.e.* we assert either that there is x that is not y, or that there is y that is not x. The expression therefore, $x \neq y$, is the simple denial of a (quantified) universal proposition; and yields an alternative of two particulars. We may expand it into $x\bar{y} + \bar{x}y > 0$.

So far we have fully discussed categoricals, but have only incidentally alluded to disjunctives and hypotheticals. This omission will be remedied in the next chapter, when it will be found that the peculiarities of the hypothetical form give occasion for considerable discussion, in the course of which the case of the disjunctive cannot fail to be involved also.

Something however must be said at once about this latter form of proposition. Disjunctives fall into two classes, according as their distinctive characteristic is displayed in the predicate or in the subject. Propositions with disjunctive predicates offer no peculiarity, for in them the subject is merely referred to an aggregate instead of to a simple class. Just as 'x is y' is expressed as $x=vy$ or $x\bar{y}=0$; so 'x is y or z' is expressed as $x=v(y+z)$ or $x\bar{y}\bar{z}=0$. And so with any number and variety of class-components in the predicate. Whether the propositions be expressed in

the affirmative or in the negative form, they differ in no essential respect whatever from the simple categorical.

When however we come to the case of disjunction in the subject we find a striking difference[1]. If we adopt the same plan as before, by making an aggregate subject of the two alternative terms, we should be prompted to express 'Either y or z is x' in one or other of the equivalent forms, $y+z=vx$ or $(y+z)\bar{x}=0$. But a moment's consideration will show that what these say is that *both* y and z are x, not that one or other of them is x. The reason why this is so lies mainly in the distinction between subsumption or predication on the one hand, and equation on the other[2]. Between 'x equals $y+z$', and '$y+z$ equals x' there is, of course, no difference whatever: we are simply equating one class to the sum of two others. But between 'x lies in $(y+z)$' and '$(y+z)$ lies in x', there is more than a difference in the mere facts stated. In the former case we are taking a single class and referring it somewhere or other within the combined range of two others. In the second case we are taking two classes together, and referring them *both* somewhere within the range of a third. The case of 'x *equals* $y+z$' is the intermediate one between these two subsumptions; *i.e.* instead of either being included in the

[1] This is well emphasized by Prof. Schröder (*Vorlesungen*, I. 341).

[2] The student of Boole may remember that he has (*Laws of Thought*, p. 171) expressed the two propositions: 'If x is true or y is true, then z is true', and 'If x is true, then either y is true or z is true', by the respective formulæ, $x\bar{y}+\bar{x}y=v\,.\,z$, $x=v\,(y\bar{z}+\bar{y}z)$. But then x and y do not here represent (as with us) *classes*, but *propositions*. He regards the terms as mutually exclusive, and capable only of the values 0 and 1. Under these conditions, paradoxical as it may seem at first sight, '*x and y*' and '*x or y*' as subjects coincide in signification: or rather a single symbolic expression will be verbally read off, in the one case with an 'and', and in the other with an 'or'. We shall recur to this in a future chapter.

other the two are equated. The simple explanation of the fact,—puzzling as it may seem to some persons, at first sight,—why there should be this marked difference between disjunction in the predicate and in the subject, lies in the fact that in ordinary universal affirmatives the subject is, and the predicate is not, distributed. There is no corresponding difficulty in the case of negative disjunction. We express, 'Neither y nor z is x', and 'x is neither y nor z', by the equivalent forms, $(y+z)x=0$, $x(y+z)=0$.

The former of the two affirmative forms is a true disjunctive. What we assert is that, collectively, the whole of x is somehow distributed between y and z; and that, individually, every single x must be either a y or a z, possibly both. The usual disjunctive inference therefore follows, viz. that every x which is not y must be z, and that if the whole class x is extruded from y it must lie within z. No such inference, of course, can be extracted from the other form, which states positively that every y as well as every z is an x.

The question will naturally be asked here, how then do we propose to represent propositions with really disjunctive subjects? The answer that we cannot (generally speaking) do this, will at first sight seem a confession of utter failure to follow the requirements of the simplest operations of ordinary thought. All however that is meant by saying this is that we cannot do it in the same way as before, viz. by displaying the actual class relation underlying the proposition; the relation, that is, of which the verbal statement is the outcome and expression. This is not a failure in our system, but springs from the facts of the case. There *is*, generally speaking, no such relation involved: there is nothing corresponding here to the subsumption of x under y in 'All x is y', or of x under $y+z$ in 'All x is y or z'.

The fact is, I apprehend, that propositions of this description are best regarded as two distinct propositions with one common predicate. If so, it is best to regard these propositions as two distinct elements, and to symbolize them by separate letters,—say α and β,—and to express the relation by $\alpha+\beta=1$, representative of the fact that these two cases cover the whole range of possibilities then and there contemplated. This view of the question will recur in a future chapter.

In support of this peculiar treatment it must be noticed that propositions of this latter kind do not, as a general rule, conform to the assumptions we have laid down concerning the import of the subject terms of categoricals. The majority of their subjects are individuals, and in almost every suitable instance which occurs to me the existence of both the alternative subjects is taken for granted. It would generally make nonsense of the statement to interpret it as signifying 'Either x or y, if they exist, is z if such there be'. The implied conditions seem to me, as a rule, to forbid such an assumption, and therefore to put the propositions on a different footing. The symbolization of each proposition, as a whole, by a separate letter, meets this difficulty; for, as will be seen hereafter, the putting $\alpha=0$ simply indicates that the proposition is false, not that its subject or predicate is non-existent.

The above remarks apply to universal disjunctives: in the case of particulars (which are not however very common) there does not seem to me to be any distinction of the above kind. Whether we say 'Some x is y or z', or 'Some y or z is x', we are alike declaring that something within the range of $x(y+z)$ is to be saved. 'Some lodging-house keepers are either much maligned or are rather thievish', 'Either some of the foremen or some of the workmen have been

very negligent':—it seems quite in accordance with popular convention to regard both of these statements as declaring that some instances are actually to be found which fall within the ranges thus indicated.

It must be clearly understood that what we have been proposing to do in this chapter is to start with the propositions of ordinary life and the common logic, and to express them in symbols. With this object in view we naturally began by selecting expressions which should preserve some analogy of form with the original verbal statement, according as that was affirmative or negative, universal or particular, and so forth. But the reader must recognize that there is no necessity consistently to adhere to these distinctive expressions. The mere fact that we put x and $\bar{x}$ upon the same footing, as ordinary class terms, reduces to comparative insignificance many of the differences of verbal statement. If, for instance, we find the expression $xy=0$, there is no reason why we should not with equal readiness read it off as 'No x is y', 'All x is $\bar{y}$', or 'All y is $\bar{x}$'. And each of these renderings again may be thrown into either a collective, a distributive, or an individual form:—*e.g.* Class x is distinct from class y: none of the x's is a y: if anything is an x it is not a y:—Class x is included in class $\bar{y}$: every x is a $\bar{y}$: if anything is an x it is a $\bar{y}$:—Class y is included in $\bar{x}$: every y is an $\bar{x}$: if anything is a y it is an $\bar{x}$.

This multiplicity of available equivalent expressions becomes much more prominent when we introduce more than two terms, as then the separate elements of the resultant propositions will frequently themselves break up into propositions. Take, for instance, the expression $xy\bar{z}=0$, the simplest rendering of which, and the one most closely in accordance with the assigned form, being, 'There is no $xy\bar{z}$'. But all the following are equivalent and legitimate renderings,—

Simple Categorical: affirmative and negative	All xy is z All $x\bar{z}$ is $\bar{y}$ All $y\bar{z}$ is $\bar{x}$	No x is $y\bar{z}$ No y is $x\bar{z}$ No $\bar{z}$ is xy.
Disjunctive	All $\bar{z}$ is $\bar{x}$ or $\bar{y}$ All y is z or $\bar{x}$ All x is z or $\bar{y}$.	
Hypothetical (negation)	If there is no zxy, then No x is y .... $\bar{y}x\bar{z}$, ... All x is z .... $\bar{x}y\bar{z}$, ... All y is z.	
Hypothetical (affirmation)	If there is xy then there is z .... $x\bar{z}$ .... $\bar{y}$ .... $y\bar{z}$ .... $\bar{x}$.	
Disjunction	Either there is zxy or No x is y .... $\bar{y}x\bar{z}$ ... All x is z .... $\bar{x}y\bar{z}$ ... All y is z.	
Limitative	Of the x's each y is a z Of the y's each x is a z Of the $\bar{z}$'s each x is $\bar{y}$: or, No x is y.	

Several of the above renderings,—the hypothetical in particular,—will presently give rise to considerable discussion, but the student should thoroughly familiarize himself with the fact of their equivalence. It is one of the merits of the symbolic rendering that, from its symmetrical arrangement, this equivalence is intuitively obvious in many of the above cases, and can be readily verified in the others: whereas some of the verbal renderings have ramified so far apart that it is not easy at once to recognize their actual equivalence.

These last remarks will be expanded and enforced in the next chapter, but it must be insisted here that they go to the root of the whole symbolic system. We say, for instance,

that 'Either there is xyz, or all y is z', is equivalent to $\bar{x}y\bar{z}=0$. By this we do not mean that the verbal proposition is necessarily *the* rendering, but that it is *a* rendering of the symbols. Given the latter, the former is one of a number of alternative equivalent renderings. Given the former, the latter is necessarily assigned; for it is the minimum relation,—that is, with our rendering of the universal, the minimum amount of destruction,—which will justify it. This last condition is important, for otherwise the symbolic statement would not be determinate. We do not say, for instance, that the destruction in question ($\bar{x}y\bar{z}=0$) is the only one which will justify us in asserting that 'either there is xyz, or all y is z', but that it is the smallest assumption which will do so; and therefore, postulating that we assume the minimum, the condition becomes determinate. Such an explanation as this is no novel introduction at this stage: it underlies the simplest rendering of the ordinary categorical. Thus we say that $x\bar{y}=0$ stands for 'All x is y'; but this same proposition is also contained in the wider destruction given by the addition $\bar{x}y=0$. The reason why we select the former is that it is the least demand required for the purpose.

The full discussion of the consequences of combining a number of logical equations will occupy a future chapter; but one or two remarks may here be made upon the contrast in this respect between the cases of universal and particular propositions. If we take what may be called *units* of propositional import,—*i.e.* assertions dealing with only single ultimate combinations,—we find that, as we add on one such unit to another, the two classes of propositions agree in their growth of significance up to the extreme case, but depart from each other there. Suppose, for instance, we have four terms and consequently 16 ultimate combinations. Starting

with the absolute insignificance indicated by the mere framework or diagram of 16 compartments, each successive destruction, or conservation, renders our knowledge more complete and precise. But whereas all 16 may be saved, the attempt to destroy 16 leads to direct contradiction.

The above applies, of course, to the case of adding on the successive items of information by means of independent elementary propositions. If we combine them in the form of *disjunctives*, we find that the essential distinction between negative and affirmative disjunction at once makes itself felt. The former is distributive, so that the more elements are introduced the more precise our information becomes. The latter is alternative, so that with increase in the number of our elements information is rendered vaguer. As a result the attempt to introduce every alternative in the former case leads to direct contradiction, and in the latter case leads to the insignificance of a general *à priori* necessity. They are respectively the negation and affirmation of our one fundamental canon of consistency. The latter case will recur in our discussion of the problem of Elimination.

A few examples of propositions are added, in order to illustrate the use of our symbolic expressions, as explained in the course of this and preceding chapters. The reader will observe that we purposely employ sometimes one, and sometimes another, of the various alternative forms of the Universal Affirmative which were noticed at the commencement of this chapter.

1. 'Men who are honest and pious will never fail to be respected though poor and illiterate, provided they be self-supporting; but not if they are paupers'. As explained, the various particles here used must all alike be replaced by the mere symbols of connection, so that the proposition may be phrased as follows:—All honest (a), pious (b), poor (c),

illiterate (d), self-supporting (e), are respected (f); and no honest, pious, poor, illiterate, paupers (g) are respected.

$$\begin{cases} abcde\,(1-f)=0, \\ abcdgf=0. \end{cases}$$

Of course when, as here, a whole group of terms presents itself which does not demand analysis into its details, we may substitute a single letter for the group. Thus we might replace *abcd* by a single letter.

2. 'No x can be both a and b; and, of the two c and e, every x must be one or other only'. This may be written in one sentence,

$$x=x\,(1-ab)\,(c\bar{e}+\bar{c}e),$$

or in two, separately,

$$\begin{cases} xab = 0, \\ x\,(ce+\bar{c}\,\bar{e})=0. \end{cases}$$

In the one case we make a single affirmative out of the proposition; in the other we exhibit its two constituent elements of denial.

3. 'Every a is one only of the two x and y, except when it is z or w; in the former of which cases it is both x and y, and in the latter case neither of them'. This may be expressed in three sentences:

$$\begin{cases} a, \text{ that is neither } z \text{ nor } w, \text{ is } x \text{ or } y \text{ only}, & a\bar{z}\bar{w}=v\,(x\bar{y}+\bar{x}y), \\ a, \text{ that is } z \text{ is both } x \text{ and } y, & az=vxy \\ a, \text{ that is } w \text{ is neither } x \text{ nor } y, & aw=v\bar{x}\bar{y} \end{cases}$$

or, in one sentence, and without the express sign of indefinitude,

$$a=a\,\{\bar{z}\bar{w}\,(x\bar{y}+\bar{x}y)+zxy+w\bar{x}\bar{y}\}.$$

4. As an example of translating symbols into words, take the following:—

$$a+\bar{a}\,(1-\bar{c}\bar{e}).$$

Here $\bar{c}\bar{e}$ stands for what fails to be c and fails to be e, so that $1-\bar{c}\bar{e}$ stands for all that does not so fail. Hence the given expression may be read off, 'Anything which is a; or even not a, provided that it does not fail both to be c and to be e'.

An alternative symbolic statement here would be $1-\bar{a}\bar{c}\bar{e}$, (since $a+\bar{a}=1$). It might then be read off, 'All that does not fail to be at once a, c and e'.

5. 'Every member of the Committee (x) is a Protestant (a), and either a Conservative (c), or Liberal (e); except the Home Rulers (y), who are none of the three'.

$$x = x\{\bar{y}a\,(c+e) + y\bar{a}\bar{c}\bar{e}\}.$$

The best way perhaps of interpreting the symbolic sentence here is just to substitute the significant words, when it would stand:—Every member of the Committee is a member of the Committee who is not a Home Ruler, (and then he is a Protestant, and either a Conservative or a Liberal), or he is one who is a Home Ruler, and then he is neither Protestant, Conservative, nor Liberal.

6. $$x\bar{y} + \bar{x}y + z\,(xy + \bar{x}\bar{y}).$$

This may be read off, 'x or y only; or, provided there be z, both x and y or neither of them'. An alternative symbolic statement (admissible here, since the terms are mutually exclusive) would be $1-\bar{z}\,(xy+\bar{x}\bar{y})$, which might be read, 'All excepting what is not z, but is both, or neither, x and y'.

7. As an illustration of the symbolic signification of particular propositions we may take the following:—'Every ab is either x or y only, and it is known that there are some a which are x and some b which are not y'.

If the latter clauses were omitted, the sentence might be written simply:—

$$ab = \tfrac{0}{0}\,(x\bar{y} + \bar{x}y).$$

This would merely obliterate the two classes $abxy$ and $ab\bar{x}\bar{y}$, leaving the remaining 14 classes perfectly indeterminate, subject to the formal condition that one at least of them must be represented. Now the statement 'Some a is x', or, 'There *are* such things as ax', puts a check upon the destruction of ax, insisting that some one at least of its four constituents (or rather three, since $abxy$ is gone) shall be saved. But it does not tell us which of them is thus to be rescued. Similarly, 'There is b which is not y' saves some one at least of the three surviving elements of $b\bar{y}$. On the diagrammatic scheme this would be carried out by our taking a note, by aid of some distinctive mark, that the whole compartments ax and $b\bar{y}$ were not to be shaded out in any case. If to the above data we add the statements 'All b is a; No x is y'; then it will be found that the 'b which is not y' will be definitely restricted to the one subdivision, $abx\bar{y}$, which must therefore certainly be saved. We should consequently have the particular, 'Some abx is not y'. But the 'a which is x' will still remain indefinite in respect of its location as regards two of the subdivisions.

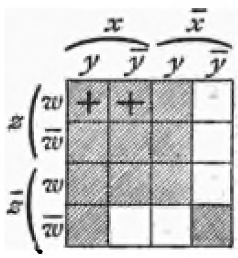
8. (i) No x is y except what is both z and w; and only some of that. (ii) Either z or w is never absent except where x or y are present, but both are always absent then. (Mor. Sci. Tripos, 1879.)

(i) May be expressed, as regards the first clause, $xy(\bar{z}+\bar{w})=0$; for 'x, except what is both z and w', means, 'x that is either $\bar{z}$ or $\bar{w}$'. As regards the second clause, we are assured of the existence of xzw that is not y, and also (I apprehend) of xzw that is y ('No members are excused attendance except those who are both old and infirm, and only some of them':—would commonly be understood to imply that some actually were, and some were not excused). If we so understand it we must write $xzwy>0$, $xzw\bar{y}>0$.

Jevons (*Studies*, p. 207) attempts to express the whole statement in the one expression, $xy = xyzwu$, where u is just another class term meant to stand for 'some'.

(ii) The first clause asserts that x, y, z, w cannot all be absent together: $\bar{x}\bar{y}\bar{z}\bar{w} = 0$. The second clause asserts that the presence of either x or y implies the absence of both z and w; i.e. xz, xw, yz, yw are all $= 0$. A material example would be given by the statement that, in some district, 'either cholera or dysentery is never absent except where there is pure air or water, but both are always absent there'. The statement, as given above, is simpler than it looks, for 'x or y' as well as 'z or w' are really treated as whole terms: each is declared to be the contradictory of the other, and each might have been represented by a single symbol.

The example is discussed by Schröder (II. 302) who regards the two propositions as meant to be combined as a pair of premises. In that case they are, as he points out, inconsistent: the particular propositions attempt to save some of what the universals destroy. This may be illustrated by a diagram.



Shade out the destroyed compartments, and we see at once that the two which are declared to be saved (ma[illegible] with a cross) have already disappeared.

CHAPTER VIII.

HYPOTHETICALS.

In the course of the preceding chapters we have had frequent occasion to touch incidentally upon the treatment of Hypothetical propositions, both in the symbolic expression of our data, and in the interpretation of our results. The subject however is too extensive and too intricate for merely incidental treatment, so a separate chapter is here devoted to a more detailed discussion of it.

As one of the principal objects of this work has been to illustrate the relation of the Symbolic Treatment of Logic, not only to the ordinary scheme but also to the current modes of popular thought, there is a preliminary enquiry which cannot be passed over. What is at bottom the significance of the familiar hypothetical form of speech? We find it of universal prevalence in all the languages of which we need take account. What far reaching conditions are there either in the phenomena of nature, or in the attitude towards them adopted by the human mind, which have given rise to this form of speech? It need hardly be said that this is not the enquiry undertaken by the ordinary logician when he lays down what *is* the signification of this or any other popular form. What he is then commonly engaged upon is

legislation rather than investigation. He is defining the sense in which he intends to use the term, or in which the best authorities have used it; and therefore consistent adherence to the sense thus adopted is imperative. But when we are enquiring into the origin of some peculiar form in common speech we cannot look for strict consistency. The utmost we can hope to do is to ascertain the dominant meaning; and, where this is apparently departed from, to detect if possible some trace of the original signification, or at least to show what is the path along which the digression has been made[1].

Briefly put, my own view as to what may be called the fundamental significance of the hypothetical form is best expressed by saying that it (1) implies a connection, of the kind called a uniformity, between two or more phenomena; and (2) implies, along with this, some doubt on our part as to the actual occurrence, in a given instance, of the pair or more of events which compose this uniformity. In other words, when we feel tolerably sure that we have got hold of a true connection between certain events, but are distinctly doubtful whether the case before us is one in point, then we naturally express ourselves in the language of hypothesis. As the latter of these two elements is perhaps the most distinctively characteristic, we will begin with it, and the reasons why the symbolic system is entitled to pass it by.

In what sense then, it may be asked, can this characteristic of doubt be considered by us as extra-logical? Why are we to regard it as giving rise to no more than an optional difference of formal statement? Simply because no attempt is made to quantify or measure our degree of doubt. Sometimes we are able to say how much doubt we entertain about

[1] I have described my own view pretty fully in my *Empirical Logic*, and therefore give only a brief outline of it here.

our propositions; that is, we are able to give some statistical details as to the proportion of cases in which our statement would be right. When this is so the proposition is at once rightfully claimed by the Theory of Probability, and the reader must be referred to that subject for the proper methods of treating it. But often the case is otherwise. We may merely entertain a vague degree of doubt which we could never venture to estimate, or, having means of estimating it if we chose, we may not at the moment wish to do so. Propositions thus entertained are what the Theory of Chances would not care to accept, and it seems to me that they form the bulk of what are commonly put into the guise of Hypotheticals. In this respect popular language seems to have rather happily seized upon a characteristic form for a distinction which, though not capable of accurate definition, and though not leading to any very serious logical distinctions, is of very considerable practical importance in the communication of our thought.

As regards the limits of this characteristic doubt, it should be observed that it refers only to the actual occurrence, under some given circumstances, of the elements which compose this connection or uniformity: it does not affect their mutual relation of invariable or general sequence or coexistence. We ought to feel tolerably sure of the connection, but feel some doubt about its individual occurrence then and there. I know, say, that a fall of the barometer will be followed by rain; this is the general regularity. But I do not know whether it will fall tomorrow; this lets in the specific doubt. If I did know that it would fall for certain tomorrow, I should only require categorical propositions. I should say, The glass will fall and it will rain. Or again, if I knew how likely it was that the barometer would fall, I might say; 'The chances are so and so that it will fall, and that

consequently (so far as this intimation is concerned) it will rain'. Here again there seems no natural opening for anything in the way of a Hypothesis. But it is when I entertain a vague degree of doubt as to what the barometer will do, that I naturally take to using an 'if' to indicate my state of mind. I express myself by saying 'If the glass falls it will rain'. Here at any rate, whatever may be found in other examples, a categorical proposition, suffused with doubt as to its present application, would seem to express all that we have to say: 'All falling barometers are followed by rain, but I don't know that the barometer will fall tomorrow'.

The two elements:—connection between the phenomena, and a degree of doubt on our part:—appear to be present in the case of every hypothetical, at any rate when it is most appropriately used; though sometimes one or other of the two may seem to be missing. The distinction therefore which is sometimes drawn between the 'if' of inference and the 'if' of doubt, does not seem to me quite accurate. There ought always to be a uniformity, giving ground to the inference, and there ought always to be a doubt whether the case before us is one in which that inference applies. But sometimes one, and sometimes the other of these two elements may seem to drop out of sight.

Sometimes, for instance, we may affect a degree of doubt which we do not really entertain, and the inferential part becomes unduly prominent. This is a perfectly legitimate rhetorical device, and will serve to bring most categorical propositions into the hypothetical form; as when I say, 'If so and so is a man he must have the feelings of a man'. Often however such a transformation as this can hardly be classed amongst rhetorical devices but represents the true functions of the hypothetical. Thus when we put 'All x is y' into the

form 'If anything is x then it must be y' we distinctly promulgate the doubt whether everything which professes to be an x is really one. That is, we are in a state of doubt as to the application of some general rule in the individual case.

It is to this latter class of cases on the whole that the hypotheticals of mathematics and the more rigid of the Physical Sciences should apparently be referred. They are plentiful enough in most treatises on these subjects, but where is the accompanying doubt to be detected? In one or two points. For one thing it is not certain that the problem or the experiment will be gone through. 'If two chords intersect within a circle the products of their respective segments will be equal'. Perfectly certain as it is that the chords will intersect one another thus if they be drawn, *will* they be drawn? it is optional with the performer to do it or not as he pleases. A general statement of the problem may fairly intimate some doubt to this effect. Moreover it must be remembered that however certain any individual may be about the circles and lines he means to draw, he is standing here in the midst of certainties of a far higher order. The glare of such formal assurance may well make any moral or physical confidence seem doubtful in contrast. Moreover there is another point which seems worth notice in reference to geometrical constructions. When I do draw my figure, what sort of a circle and chord do I obtain? one so defective in comparison with its ideal that we may well throw a doubt over the actual execution of that which was intended to be done. The depression of the one element throws into relief the height of the other. The greater the stress we lay upon the doubt, the more prominent do we render that rigid connection into which no shadow of a doubt can enter.

This connection or uniformity, as we need scarcely ob-

serve, may be exhibited in any order as to time, for the so-called 'consequent' clause in the hypothetical sentence may in reality either succeed the 'antecedent', or be coexistent with it, or be succeeded by it. All that is requisite is a reasonable degree of regularity of occurrence as between the two factors which constitute the connection. For example, starting with the doubtful fact intimated in the antecedent 'if I have a headache tomorrow', I may attach to this doubtful fact a certain 'consequent' in any order of time past, present, or future. I may go on to say, 'then I must have eaten something unwholesome today'; or, 'I shall keep to my bed so long as it lasts'; or, 'I shall be more careful as to my diet in future'.

It may fairly be enquired, why the case of these doubtful *connections* should be thus singled out? Doubtful *facts* obtain no special treatment; why should groups of facts which have a doubt about them be distinguished, both grammatically, by the appropriation of distinct linguistic forms, and also logically, by being commonly discussed in a section apart? The answer must be, because of their great importance. A doubtful isolated fact, if such a thing were found, would be of very little service to any one, unless as a hint towards further and more careful observation. But generalized connections of any kind may be of the utmost importance to us, for these constitute the basis, and indicate the limits, of all knowledge and of all sure and safe practice. It seems to me that the Hypothetical proposition, as above interpreted, precisely indicates the mental attitude of the most thoughtful and best informed persons at almost every moment of their lives, but especially at critical conjunctures. They are familiar with abundance of suitable uniformities, under the name of laws; as to which they feel, or should feel, no manner of doubt. But what they must feel doubtful about is the individual

application, the occurrence of the antecedent. Doubt here is in many cases just as desirable and scientific as is certainty there. This double element seems happily indicated by the form of the hypothetical sentence and it seems to me to be only where both of these are present that this form can be used with rigid propriety. Common language is therefore abundantly justified in making a separate class of propositions of this description, but it does not follow that Logic in general, still less Symbolic Logic,—should follow the example, and regard them as falling under any distinct rules of treatment. On the contrary, it seems to me that such characteristics as these are, so to say, strained out and left behind in passing through the framework of what is formal.

We now come to the question of the proper symbolic rendering of hypotheticals. As the reader will probably have concluded already, there appears to be no distinct form peculiarly appropriate to them :—at least not on the scheme adopted in this work[1]. We are therefore not proposing to invent a formula which shall be so peculiarly suitable to the assigned proposition that no one, on seeing the formula, can fail to read it off in almost the very terms in which the proposition had been couched. We shall be satisfied by the employment of a formula which is *equivalent* to that proposition, in respect of what it strictly asserts, denies, or leaves doubtful. On this point the practice of the Common Logic can be no guide to us. So much importance is there attached to the mere form of expression that many propositions are ranked as distinct though it is obvious to every

[1] Some schemes adopt a notation which follows much more closely the forms of ordinary verbal assertion. Thus Mr McColl writes $x:y$ for 'x implies y': from this it follows that we can write $(x:y):(z:w)$ which is naturally read off as 'if x implies y then z implies w'. Here x and y stand for, not classes, but assertions.

one that they convey precisely the same information: e.g. No A is B, and No B is A. The system in fact goes rather beyond popular convention in the importance it attaches to mere verbal form. But in the Symbolic Logic it is otherwise.

It is largely a matter of our own choice how we read off our premises and conclusions. In the symbolic statements themselves there is nothing to indicate in what way the premises were worded when they were handed over to us, nor is there anything to force us to translate them back again into one form rather than another. This fact must have presented itself to the reader from our first discussion as to the import of even the simplest form of categorical assertion, but its full significance is only seen when we come to deal with complex propositions. Take, for example, the statement $AB = ABCD$. This may be read off in a variety of ways:—All AB is CD: If any A is B then it is CD: If any B is A then it is CD: whenever an A is B then it is a C which is D: and so forth. That these different renderings involve different *judgments* is quite true; and therefore on a more subjective view of Logic they must be distinguished from each other. Also, when A, B, C, D are translated into concrete terms it may happen that common sense and grammar reject one of these renderings and insist upon another. But it has been abundantly illustrated in the course of this work that what we look to are the actual class relations involved, and therefore we are prepared to allow much latitude in respect of the way of reading off our statements, provided these relations themselves remain undisturbed.

It is quite true that logical equations which involve complex terms are more naturally interpreted in a hypothetical form than are those of a simpler kind. Indeed, in a recent

treatise on Symbolic Logic[1], the form $xy = c$ is chosen as the type of the Hypothetical, as $x = c$ is of the Categorical. It will be instructive to enquire into the ground of this choice. There is no doubt that each of these forms can, if we please, be interpreted in either way. Thus $xy = c$ may be read off (treating the left side as the subject) either 'All xy is c', or 'If any x is y then it is c'; whilst $x = c$ may be similarly read off either 'All x is c' or 'If anything is x then it is c'. Why then is the former regarded as more appropriately hypothetical? Mainly, I apprehend, for the following reason. It was shown in a previous chapter (Chap. VI.) that every universal affirmative proposition must be interpreted as involving something of a hypothesis; 'All x is c' having to be understood, if we wish to work with it successfully, as meaning 'All x, if there be any, is c'. But this hypothetical element is generally so faint as scarcely to be perceptible in common discourse, where indeed it is often entirely rejected. Names are seldom employed except to denote what we suppose to exist, so that we come to feel much reluctance to assert that 'All x is c' unless we are convinced of the existence of x. Hence any doubt on this score commonly drops out of sight, and the categorical is safely assumed in most cases to carry with it an assurance of the existence of the subject and predicate. But though such an assurance may be justified in the case of x and y separately, it is quite another thing to justify it in the case of the com-

[1] Macfarlane's *Algebra of Logic*, p. 81. Lambert, so far as I know, was the first to explicitly assign this notation for the expression of a hypothetical:—"Die allgemeinste Formel der hypothetischen Sätze ist diese: Wenn A ein B ist, so ist es C. Diese Formel kann allezeit mit der folgenden verwechselt werden; Alles A so B ist, ist C. Nun ist, Alles A so B ist $= AB$; folglich, Alles AB ist C. Daher die Zeichnung: $AB > C$, oder $AB = mC$" (*Log. Abhandlungen*, I. 128). Some explanation of the symbols thus employed will be found in the final chapter.

pound xy. These symbols, when occurring separately, stand presumably for common terms which are familiar to every one, but their combination into one may be something novel, and perhaps altogether doubtful. Hence the enquiry whether there be any such thing as the subject of the proposition, which, to common apprehension, would seem almost impertinent in the case of the simple propositions typified by $x=c$, becomes quite pertinent in the case of the complex propositions typified by $xy=c$. The doubt thus suggested naturally expresses itself by our adopting, as the verbal equivalent of the latter form, such a conditional statement as, 'If x is y then it is c', the rest of the complete statement, indicated by the symbols, taking the form 'c is xy, if there be such'. Propositions therefore with complex subjects almost force upon us that hypothetical interpretation which we have found it advisable for symbolic purposes to extend to all propositions without exception.

With these preliminary difficulties cleared out of our way we will now proceed to the discussion of the various hypothetical forms in detail. Owing to their variety and ultimate complexity we shall find it well to divide them into three classes, commencing with the simplest, as follows:—

(1) Those in which there is only a duality of elements in thought, though there may be three or four elements or terms in the expression. That is, those of which 'If P then Q' is an adequate and valid representation.

(2) Those in which three elements are distinctly involved in thought; that is, of such a form as 'If P is Q then it is R'.

(3) Those in which four distinct elements are contemplated. These are by no means common; at least not in their full generality. But they cannot be passed over in a symbolic treatment of the subject. Indeed, if we follow the

usual practice in mathematics, we ought not to be satisfied unless we can offer an expression for the most general case possible, and then show how the simpler cases can be deduced from this by the requisite modifications or omissions.

I. We have first to consider the cases in which the hypothetical merely connects two simple elements, or elements which may be treated as simple; that is, cases of which the form 'If P then Q' is an adequate representation. In thus speaking of simple elements we must once more remind the reader of the extent to which this simplicity is at bottom conventional, or dependent upon linguistic conditions.

Sometimes this simplicity, and the appropriateness of the form 'If P then Q', will be obvious in our very language. This is especially the case when we are dealing with impersonal verbs. Thus 'If it lightens it will thunder', obviously contemplates two elements only, lightning and thunder, and regards one as dependent upon the other. It is therefore fitly represented by the simple form 'If P then Q'. This simplicity is however closely connected with the particular form of speech employed. A language which did not happen to possess these impersonal verbs might be obliged to break each clause up into two parts, and express each by means of a subject and predicate, as by saying 'If the sky flashes fire then the air will give a crash'; or something to that effect, so as to give the semblance of four distinct elements.

This essential simplicity of conception exists in many more cases than might at first be supposed. The impersonal verbs at our command are comparatively few, and we are often obliged in consequence to build up, out of a combination of terms, conceptions which are nevertheless really regarded as simple elements for the purpose in hand. Again,

this is prominently the case when we are concerned with events which concern the action of individuals, and for which therefore we are not prepared with a current term. Thus when I say 'If a Government resigns office the Queen will send for the leader of the opposition', this is naturally, and indeed inevitably, expressed in the complex form 'If A is B then C is D'. But we are surely only contemplating two elements here just as with the thunder and lightning; though since it happens that these elements have not acquired single descriptive terms to indicate them, we find ourselves obliged to build up the requisite designations by a combination of two or more simpler terms.

Again, there is another somewhat different class of cases to which the same sort of explanation will apply; that is, in which an apparently complicated proposition, *i.e.* one involving four logical terms, is essentially simple in its character and only involves two distinct contingencies. For instance, we often employ the quadruplex expression 'If A is B then C is D' in cases in which A and C are supposed to be taken for granted, and the only doubtful elements are the existence of B and D as modifications respectively of A and C. Inasmuch therefore as the disappearance of A and C is out of the question, the antecedent and consequent each practically consist only of a single element, so that the whole expression may be adequately represented by the simple form 'If P then Q'. Thus for example when I say, 'If Cambridge wins the boat race I gain a sovereign', it is clear that the non-occurrence, so to say, of Cambridge or of myself is not contemplated. They are taken for granted as present throughout, so that the only possible antecedent is Cambridge winning or not winning, and the only possible consequent is myself gaining or not gaining. That is, though the logician may adhere to common speech, and throw such a

proposition into the form 'If A is B then C is D' it belongs essentially to the simpler class 'If P then Q'.

I call attention to this point here, for, as we shall presently see, true quadruplex hypotheticals do actually exist, or at least can readily be invented. What we require however for the full generality of such a form as 'If A is B then C is D' is something more than the mere occurrence or non-occurrence of 'A as B' and 'C as D'. We should have to contemplate the possibility of all the alternatives included by the various assumptions that there is B without A, or D without C, or that there is no A or no C. In a word, A, B, C, D must be true logical class terms with the full privileges of such terms at their disposal.

The conclusion then that we have so far come to is that a very large proportion, in fact an overwhelming majority, of ordinary hypotheticals, though they may demand for their convenient expression more than two terms, are nevertheless to all intents and purposes composed of a pair only of elements. Their actual complexity is mainly a matter of linguistic propriety or convenience. They admit therefore of adequate verbal expression in the form already alluded to, viz. If P then Q.

The only remaining question therefore here is, How should one of these hypotheticals be represented in symbols or in diagrams? Or, in accordance with the requisitions laid down at the commencement of this chapter, what definite class relation between P and Q is necessarily involved in order to give rise to this hypothetical, and to it (or one of its equivalents) only? We answer simply that the requisite condition is that there shall be no P without Q; i.e. that $P\bar{Q}$ is non-existent, ($P\bar{Q} = 0$).

It need hardly be pointed out that all which we are thus doing is to express the categorical proposition 'P is Q': and

this ought of course to be so, on the interpretation here adopted. If the subject of the proposition, P is Q, is not taken for granted, then this proposition differs in no respect beyond that of mere phraseology from the hypothetical, 'If P then Q'. In other words, any hypothetical which substantially only involves two elements may be regarded as equivalent in signification to a categorical affirmative involving those two elements, and as having been sufficiently discussed already under that head.

There is one difficulty, which will naturally suggest itself here. It may be urged that there is a broad distinction between the cases which we thus identify: that in 'P is Q' we have true predication, viz. the assignment of a predicate to a subject, whereas in 'If P then Q' we may have merely two events, possibly far removed from each other. This is true; and in many cases it would be very misleading to call the latter relation by the name of 'predication', unless we had previously pointed out the generalized significance which Logic should impose upon this term. It must be remembered then that however nearly simple and simultaneous may be the facts,—the substances and attributes,—which the more elementary forms of proposition are employed to connect, yet there is a strong tendency in language to embrace more and more complicated abstractions into the synthesis of a single term, and to connect more and more remote events, by means of these terms, into the synthesis of a single judgment. Thus when we say that 'every child is mortal', or that 'Hydro-pyridine[1] is poisonous,' we are using the same form of predication as when we say 'the stone is heavy'; but in the first case we are bringing into relation events which may be many years apart, and in the

[1] A laboratory synthesis: suggested to me by a chemist, as having been extremely seldom produced and pretty certainly never swallowed.

second we are contemplating a result which may never be a matter of experience. In fact in this latter instance if we were asked to explain what we mean we should almost inevitably fall into the hypothetical form by saying that if any one did swallow the substance he would die. To say that the child *is* mortal now, and that hydro-pyridine *is* poisonous, though the one contemplated event may not happen for many years, and the other probably will not happen at all, is to surrender to a device of language. The whole significance of the terms is remote or problematical, and therefore to employ them as predicates is, at least, to antedate their real reference.

Terms which in this way antedate a remote event, and thus yield the semblance of present predication are not common, even in the case of familiar objects. And when the antecedent and consequent are events of only occasional occurrence, suitable expressions for this purpose cannot be expected. Thus if P stands for 'Lord Rosebery going out of office' and Q for 'The Queen sending for Lord Salisbury', it may appear as if this was necessarily a case of 'If P then Q', and not of 'P *is* Q'. And yet the consequent is not really more remote than some which the aid of a convenient verbal expression enables us to bring into such close relation with the antecedent as to give them the standing of predicates, and thus to make them rank as attributes.

Briefly put, my meaning is this: I fully admit the broad practical distinction between the cases which fall naturally into the 'If P then Q', and those which fall into the 'P is Q' form; and also the fact that although common speech can sometimes bridge over the distinction by the convenient introduction of a predicate, it far more often cannot do this. But none the less they both display the same essential characteristics of what may be called 'predication' in a

generalized sense of the term. They both alike yield the four alternatives which we want: viz. *P* asserted, *Q* asserted: *Q* denied, *P* denied: *P* denied, *Q* (so far) doubtful: *Q* asserted, *P* (so far) doubtful. And this is the essential point for purposes of judgment and inference.

II. We will now proceed to discuss the next stage of complexity, viz. that in which not only three terms are involved in the verbal statement, but in which three elements, viz. classes or class terms, are explicitly present in thought. It is necessary to introduce this latter condition, because a hypothetical with nominally three elements may of course, as in the instances above referred to, involve no more than two distinct elements in thought. When I say 'If *A* is *B* then it is *C*', I may have in view a case in which the only contingencies are those of *A* being *B* (or not), and *A* being *C* (or not), whereas what we now assume is that we may have to take account of all the contingencies created by the possible combinations of *A*, *B*, and *C*. That is, *B* and *C* must each be a class term, and *A* also a class term or an individual, and we must be prepared to take account of all the logical sub-classes which these conditions furnish.

The question before us, then, is this:—What is the underlying class relation which will find adequate expression in some assigned hypothetical proposition? We might begin from either side of this correlation. That is, we might begin by drawing up a schedule of all the possible hypothetical forms, and then proceed to examine what class relation is exactly implied by each of them; or we might start with the class relations themselves, and examine what verbal propositions would adequately express them. Either of these processes would be tedious, if completely carried out. We will therefore confine ourselves to a mere selection, and will adopt the second of the above courses.

1. Suppose then that one element only is obliterated, say $AB\bar{C}$. This appears to me to be adequately expressed by the hypothetical, If any A is B then it is C; or by, If any B is A then it is C. Or again, if we prefer a negative form, we may express it by saying, If there is no AB that is C then no A is B: a statement which itself admits of several verbal modifications. A concrete example would be, If any ratepayer is a publican then he is against the bill. This hypothetical phraseology implies exactly the same relation between the classes involved as the categorical denial that there are any such persons as 'publican ratepayers not against the bill'.

It will be seen that we have phrased our hypothetical above in the form 'If *any* A is a B'. It will therefore follow that 'If *all* A is B then all A is C'. That is, we may seem to have got the categorical equivalent for the hypothetical, commonly expressed in the form If A is B then it is C.

This was, in fact, the view which I took in the first edition of this work. But further reflection, aided by the criticism of Mr Johnson, has shown that although there is nothing incorrect in this rendering, it cannot strictly be called adequate. It is quite true that the hypothetical, If all A is B then all A is C, requires the categorical $AB\bar{C}=0$; and that this is the minimum requirement for its justification. If therefore we start with the former member of the correlation the categorical equivalent is unambiguous. But remark that the antecedent of the hypothetical asserts more than is necessary. It is not requisite to say so much as that if *all* A is B, all A is C: it is sufficient to say, as above, If any A is a B. All therefore that we can claim is that if we are to select a hypothetical containing only the simple terms A, B, and C, and this is to be couched in the

ordinary form familiar in Logic, then the nearest we can choose is, If A is B then A is C.

In adopting this view of the interpretation of the hypothetical it must be remembered that a similar explanation has been repeatedly offered in Logic. I do not mean by this that it has been claimed that any such precise transformation of the hypothetical into the categorical is possible, but that it has been attempted to decide what is the underlying class relation implied. Thus, 'If A is B then it is C'; has been declared to imply that 'All B is C'. It is quite possible that this relation is what the speaker may have in view; but it is clear that it is not the relation which is actually implied, if by this we mean the minimum condition demanded in order to satisfy the hypothetical. It is not necessary as we have seen, to claim that All B is C, it is sufficient to claim that All B which is A is C.

A concrete example will make this plain. Suppose some one had said in 1852, 'If Louis Napoleon becomes emperor he will be crowned', it is quite likely that it would be replied 'You mean that all emperors are crowned'. But a moment's consideration will show that though this general proposition may be really true as a matter of fact, and is, in that case, very likely to have been 'meant', yet it is not implied in the hypothetical nor is it therefore a correct rendering of it. All that is fully justified is the narrower statement which has to be put in the words 'Napoleon-emperor is (or will be) crowned'. The distinction is seen at once by varying the example. Had one said 'If Napoleon becomes emperor he will be perjured', it must not be assumed that all emperors are in that position, but merely that the individual in question is, which we may put into the words 'Napoleon-emperor is perjured.'

As this view is not familiar to logicians it may be well to

take in order the different cases which may arise by permutations of A, B, and C.

1. If A is B, A is C. As we have seen, when this is interpreted, If any A is B it is C, the strictly equivalent categorical rendering is $AB\bar{C}=0$. When we interpret it If all A is B then all A is C, we can only say that this categorical relation will justify the hypothetical and is the least which will do so. But the two cannot be said to be equivalent, because the hypothetical is needlessly wide in respect of its antecedent.

2. If A is B, C is B. The minimum class relation which will justify the rendering, If all A is B then all C is B, is $\bar{A}\bar{B}C=0$. Given this as a condition it necessarily follows that the mere fact of securing that All A is B will secure that All C is B. And it is the minimum condition which will secure it. A concrete example would be, If all the tories are against the veto bill then all the publicans are.

Here, as in the last case, it is not unlikely that we shall be told that another condition, wider than the one assigned above, is really meant, viz. that all the publicans are tories. It may be so, but this ought not to have been meant, if what was sought for was the minimum condition requisite to secure the result. It is not necessary for this purpose that All C should be A, but only the C which is not B.

3. If A is B, B is C. It will readily be seen that no class relation can be assigned here such that the mere process of securing that All A is B shall *ipso facto* secure that all B is C. The hypothetical must be regarded as one of two terms only, the fact of 'A being B' and 'B being C' corresponding to the P and Q respectively, of our first case.

This is the only result we could expect. On our interpretation, 'A is B' simply removes the A that is not B: it is

clearly impossible to provide a class relation such that this removal should by itself remove the class '*B* that is not *C*'. I am quite aware that examples may be suggested which will seem at first sight at variance with this result, but it will be found that their validity turns upon a tacit and perhaps illicit conversion of the antecedent. Thus it might be said, If Francis was Junius, Junius was a traitor: but here, Francis and Junius being singular names, the proposition connecting them may be simply converted, and we have case (1) repeated. On the other hand take such a proposition as this, If the radicals are statesmen then statesmen are humbugs. What such a speaker probably means is that this would be the inference if the radicals and the statesmen are the same class. What he is arguing from is an identity rather than a universal affirmative, and the really effective premise is not the one proposed but its universal converse, If statesmen are radicals. Only in such a way can we give a class explanation of the kind in question to such hypotheses as these.

$$\begin{cases} 4. \quad \text{If } A \text{ is } B,\ C \text{ is } A. \\ 5. \quad \text{If } A \text{ is } B,\ B \text{ is } A. \end{cases}$$

Both of these stand in the same general position as that just considered. We may couch them in the form of hypothetical inferences, but no class-relation can be proposed from which the consequent shall 'follow', in the ordinary sense of that term, upon the admission of the antecedent. We must regard them as being either two-term hypotheses, or as deriving their validity from the identity of subject and predicate.

6. There is only one remaining case which could be called for, viz. 'If *A* is *B*, *A* is *B*'. It is merely noticed as a reminder that a purely formal truth can have no significant class expression corresponding to it. The hypothesis in

question, being really insignificant, demands and can be satisfied by no class-relation whatever. It must always be true, or rather can never be false, and therefore no pre-existing relation between the classes A and B can give it any help.

Gathering up the above various cases we can readily divide them into three classes. The general object before us in all three is the same, viz. to determine what is the exact nature and amount of the implied class-relation which is involved by the mere statement of the given hypothetical; but this may vary within considerable limits.

(i) The *whole relation* demanded in order to satisfy the consequent may be contained in the antecedent, as in Ex. (6) ('If A is B, A is B'). The proposition is then insignificant, or formally true. No pre-assigned relation amongst the classes is required in order to make the hypothetical true, and it therefore admits of no symbolic representation.

(ii) *A part of the relation* thus demanded by the consequent may be contained in the antecedent, as in Exx. (1) and (2). This seems to be the most appropriate and significant employment of these hypothetical forms. The consequent is partly contained in the antecedent, and therefore the hypothetical is not ineffective; but it is only partly contained, and therefore the hypothesis is not insignificant. The necessary relation amongst the classes which is implied in the hypothetical proposition is that part of the consequent which is not contained in the antecedent. Thus in 'If A is B then it is C' the relation is, that AB shall be C; ($AB\bar{C}=0$). This relation amongst the class terms at once appropriately expresses the significance of the hypothetical.

(iii) *No part of the relation* demanded by the consequent may be contained in the antecedent, as in Exx. (3), (4), (5). The antecedent does not in that case contribute at all to the

consequent, which must therefore hold true independently, and again the hypothetical (in its ordinary interpretation at least) becomes robbed of its significance. Thus in 'If A is B, C is A', no relation can be found amongst the classes involved such as shall make C to be A, if and only if, A is B. C must be A independently of such a relation.

All this is perfectly in accord with the principles of the Symbolic Logic, or rather with those of any system which deals with the mutual relations of classes in the way of extension. What every proposition is there supposed to assert, or to imply, is the actual relation of the classes denoted by the terms involved. This is the meaning or significance of the proposition in whatever form, whether hypothetical or categorical or disjunctive, it may happen to be couched.

In other words what we have been saying above may be put thus. The hypothetical form may be regarded as an indirect mode of conveying categorical information, and it is in this categorical intimation that its significance is to be sought. Start with our complete scheme of possibilities, and introduce a purely hypothetical or supposititious restraint in the way of destroying or conserving some class. We are carried no further. Nothing fresh, that is positive or categorical, follows. I may say, 'Suppose xyz destroyed (or conserved)' and nothing more can be deduced but a similar supposal of the destruction (or conservation) of 'some' x or y, or z. But introduce a categorical element, i.e. impose the material restraint upon our formal possibilities which is implied by any kind of proposition, and the case is very different. Assert, for instance, that 'All xy is z' ($xy\bar{z} = 0$) and it would at once follow that *if* xyz were destroyed then 'no x is y'; and that *if* 'some x is y' then xyz must be conserved. That is, the hypothesis, though verbally making only a *supposal* of some alteration (in the way of destruction or conservation) is

really an alternative mode of indicating or implying a certain underlying *categorical* relation (of the same kind). This categorical relation, in its minimum of significance, is what we here regard as the import of the Hypothetical.

III. The kind of proposition next to be considered is that in which four bonâ fide class-terms are introduced: that is, propositions in which the employment of four terms is not a mere linguistic device for securing what is essentially a simple contingency,—as in the cases first considered,—but in which all the four terms have to be taken separately into account, and are supposed to enjoy their usual possibilities of combination. It is in fact an extension to four class-terms of the case just discussed, in which only three were involved.

The best plan of procedure therefore will be to begin upon the same track as before. We saw that some logicians, recognizing generally the same object as we have in view, had proposed 'All B is C' as the requisite class-relation for, 'If A is B then it is C'. But this we found to be a wider assumption than was necessary, the true requisite being the narrower relation 'B that is A is C'. The true analogue to the broader statement, where four class-terms were involved, viz. 'If A is B then C is D' would be, All C is A, All B is D. The assumption of these two therefore would precisely correspond to the single assumption in the former case; that is, they would be sufficient, in fact more than sufficient, to justify the hypothetical in question.

There is no difficulty in finding an example which shall give full scope to all the various capacities of our class-terms and thus fully justify the formula. For instance the hypothetical, If Protestants are heretics then Lutherans will be lost, is satisfied at once by the two propositions, All Lutherans are Protestants, All heretics will be lost. And we should have exactly the same sort of right to say that such was the

meaning of this hypothetical as to say that, as in a previous case, 'All B is C,' is the meaning or implication of, 'If A is B then it is C'.

The precise minimum class-relation demanded in order to satisfy our hypothetical in this case is as follows:—All C that is not D is A that is not B. [$C\bar{D}\,(1 - A\bar{B}) = 0$; or, in more detail, $CD\,(AB + \bar{A}B + \bar{A}\bar{B}) = 0$.] Grant the existence of this relation and the hypothetical follows from it.

If we select a suitable example,—that is, one in which we are not merely considering a pair of alternatives, but in which all the four class-terms can be freely combined or separated according to the fullest capacity of such terms,—we shall find that it fits in conveniently with our explanation. Suppose for instance that, amongst the inhabitants of some town, we were to take into account, (A) the householders, (B) those who are married, (C) the guardians, (D) the males; and the respective contradictories of these. Here any combination of these elements can be intelligibly joined with any other, and any one of the resultant classes can be supposed to be retained or to disappear. And suppose also, as a result of observation, that it is discovered that, All the guardians who are women are householders who are unmarried:—$C\bar{D}\,(1 - A\bar{B}) = 0$.

What we maintain is that this statement is precisely equivalent in its implications to,—that it gives the same amount of categorical information, neither more nor less, as,—the four-term hypothetical, 'If all the householders are married, then all the guardians are men' (If A is B then C is D)[1]. But, as before, we do not assert that the hypothetical is the minimum supposition which will give this information.

[1] The interpretation here suggested is somewhat different from that adopted in the former edition. As stated above, further reflection,

We may test this by beginning with either the antecedent or the consequent, viz. by examining the hypothesis, according to the common technical language, either constructively or destructively. From the former point of view what both alike imply is that the mere fact of destroying '$All\ A$ that is not B' shall ipso facto destroy 'All C that is not D'. From the latter point of view what both alike imply is that the mere fact of securing that there exists 'C that is not D' shall ipso facto secure the existence of 'A that is not B'. These two valid inferences are all which we can rightly claim of such a formula, and this requirement it yields us. It does not yield the other two well known invalid inferences.

The above results may be phrased in a slightly different way. We are told that, under a certain condition or limitation, 'C is D': what further assumption must we make in order that 'C is D' shall hold good certainly and unconditionally? If this question were proposed to the common logician his answer would be prompt. He would say, Combine the two propositions, 'If all A *is* B, C is D', and 'If some A is *not* B, C is D'; and then since one of these two alternatives must certainly hold good we are forced to the conclusion that C is D. But to take this view is to start from a different conception of the nature of the universal proposition from that which alone adapts itself to symbolic usage. Holding, as we here do, that every universal proposition simply denies a certain class-combination we make a distinction between universals and particulars which gives a very different signification to such a combination as the above.

The way we have to look at the question is this. We want to secure that All C is D, that is, that the whole of the class $C\bar{D}$ shall be destroyed. Now the necessary condition,

and the criticisms of Mr W. E. Johnson in *Mind* and elsewhere, have shown that my previous explanation was insufficient, if not incorrect.

implied in the statement of the hypothetical, destroyed, as we saw, a certain portion of $C\bar{D}$, viz. all except the $A\bar{B}$ part of it. Accordingly, if we take care to destroy this supplementary part of it, we shall have secured its complete removal. That is, the proposition expressing this last demand, when combined with the original hypothetical, will secure (and no more than secure) the absolute unconditional truth of the statement, All C is D. But the expression for this proposition ($C\bar{D}A\bar{B}=0$) is simply given in the form, No A that is not B is C that is not D.

Hence whereas, on the common view, the requisite pair of propositions is,

$$\begin{cases} \text{If All } A \text{ is } B \text{ then All } C \text{ is } D, \\ \text{If Some } A \text{ is not } B \text{ then All } C \text{ is } D; \end{cases}$$

on our view the requisite pair may be thus stated,

$$\begin{cases} \text{If All } A \text{ is } B \text{ then All } C \text{ is } D, \\ \text{No } A \text{ that is not } B \text{ is } C \text{ that is not } D; \text{ or No } A\bar{B} \\ \quad \text{is } C\bar{D}. \end{cases}$$

This will be more clearly seen if we generalize somewhat, by considering all the possible alternatives which may form the antecedent of the hypothetical. The common view only admits of two of these,—'If A is B', and 'If A is not B',—; for it takes the existence of A for granted, and thereby reduces the hypothetical to the simple two-term form, If P then Q. But the symbolic view, not taking this for granted, finds that it must admit *four* alternatives, in accordance with the schedule of propositions on page 190. These four, stated in ordinary language, are as follows:—

If All A is B, C is D or $C\bar{D}(1-A\bar{B})=0$.
If No A is B, C is D or $C\bar{D}(1-AB)=0$.
If All B is A, C is D or $C\bar{D}(1-\bar{A}B)=0$.
If 'All' is either A or B, C is D or $C\bar{D}(1-\bar{A}\bar{B})=0$.

The first thing that will strike the ordinary logical

reader as unfamiliar here is the fact that *any* pair of these alternatives will secure that C is D. This is, of course, a wide divergence from the common view, but it follows consistently from our interpretation; when, as here, we are dealing with four terms. Consider, for instance, the first and third of the above propositions. The obvious customary objection to regarding these as sufficient to constitute a complete logical dilemma is that they do not cover the whole ground between them: i.e. that there is no reason why one of the two,—All A is B, All B is A,—must necessarily hold true. Quite so, on the common interpretation; but not here. Bear in mind the three facts (1), that we are interpreting 'A is B' as simply obliterating the A that is not B, (2) that we are proposing to deal with the kind of hypothetical which involves four true class-terms; and (3) that in 'representing' a hypothetical what we propose to do is to display the minimum class-relation implied by it: and the above conclusion seems necessarily to follow. All that we say is that the class-relations thus demanded by any two of the above four propositions are sufficient to secure unconditionally that C is D.

This conclusion may be further illustrated by examining, from the same point of view, the kind of converse of the hypothetical which the Common Logic terms 'destructive': i.e. that which follows if instead of asserting the antecedent we deny the consequent. Turn to the first of the four propositions in the above table. What it asserts is that if $A\bar{B}$ be destroyed the destruction of $C\bar{D}$ certainly follows; the destructive converse of this is yielded by saying that if $C\bar{D}$ is secured from destruction so is $A\bar{B}$. With us, this stands for, 'If some C is not D, then some A is not B'. The corresponding four-fold table, from this point of view, would consequently be:

If Some C is D then Some A is B	$CD(1-AB)=0.$
If Some C is not D then Some A is B	$C\bar{D}(1-AB)=0.$
If Some D is not C then Some A is B	$\bar{C}D(1-AB)=0.$
If Some not-C is not-D then Some A is B	$\bar{C}\bar{D}(1-AB)=0.$

Several peculiarities will be observed here at once. For one thing it will be seen that, on this scheme, a universal in the antecedent demands a universal in the consequent, and similarly with particulars. This depends upon the fundamental distinction as to the import of these two kinds of propositions, one being primarily destructive and the other conservative.

It is easy enough to make such a pre-existent arrangement of our classes that the destruction (or salvation) of some assigned class-group shall destroy (or save) some other, —not inconsistent,—group. But, except in one extreme case[1], it is impossible to make such an arrangement that the destruction of one group shall *save* another; and it is impossible in any case to make such an arrangement that the salvation of some one group shall destroy some other.

We cannot, for instance, express such hypotheticals as 'If Some A is B then No C is D', 'If No A is B then Some C is D'. Is this a defect of the method? I think not, regard being had to what we actually propose; viz. to determine the class-arrangement which, on our interpretation of propositions shall necessitate such a consequence. The thing cannot be done; and if such a consequence is to hold good it must be from some other ground than that of a class-relation, in which case the antecedent and consequent

[1] I allude, of course, to the application of our fundamental formula which equates to unity the sum-total of all our classes. If we destroy all but one class, that one must survive: if we destroy any assigned number, *one or more* of the others must survive. But unless the group to be saved is the complete supplement of that which is destroyed no such inference will hold good.

must be regarded as wholes, and then we fall back upon the simple form, If *P* then *Q*.

Again; it will be seen that all the four propositions are required in order to secure unconditionally that *A* is *B*. This also depends upon the interpretation of the particular. Each of the four antecedents contributes something towards saving the '*A* that is *B*',—i.e. saves a portion of it—: the co-operation of the four is required to save the whole.

The general conclusions which have been so far arrived at, may be summed up as follows. When we seek to express the hypothetical under the proposed conditions,—viz. to display the actual class-relations involved in its symbolic representation,—we find that it is necessary to commence by a recognition that these propositions belong to two somewhat distinct classes.

I. Many hypotheticals,—in fact the vast majority of those which present themselves in common speech and in the logical text-books,—really consist of only two elements in thought: an antecedent and a consequent: though linguistic requirements may involve the employment of four distinct terms in order to express them. In all such cases the hypothetical is essentially of the form, 'If *P* then *Q*'. This we regard as being equivalent to, or rather as being covered by '*P is Q*'. In other words, the really simple hypothetical does not (on our rendering) differ in any way from the categorical, nor therefore does it demand any peculiar form of expression. Of course it may often happen that the predicative term '*is*' in '*P* is *Q*' may make nonsense according to the common conventions of language: all that we then have to do is to interpret it by the substitution of some such phrases as; *P* is a case of *Q*, a sign of *Q*, or so on. But the substantial meaning is always the same, viz. 'Without *Q*, no *P*'.

II. In a complete treatise, however, we must take into account a certain class of hypotheticals in which three, or even four terms, are distinctly involved. For instance, the expression 'If A is B then it is C', will often involve three real class-terms; and there is no difficulty in inventing concrete examples of 'If A is B then C is D', which shall introduce four such terms with all their possibilities of logical combination.

In regard to these truly complex hypotheticals some difficulty may often be experienced in determining exactly what they signify, but our general object is always the same. After determining their signification, express by symbols or by diagram the actual class-relation which is involved in them, and from which they may be inferred, and conversely, if possible. For instance, the proposition, 'If A is B then C is D', in which A and C are unquantified may possibly mean either of two things. It may be intended to signify, 'If *all* A is B then all C is D'; in which case, being a universal, we express it, 'All $C\bar{D}$ is $A\bar{B}$', $\{C\bar{D}(1-A\bar{B})=0\}$. Or it may be intended to signify, 'If some A is B (i.e. if there is any AB) then some C is D (i.e. there is some CD'), in which case, being a particular, we express it 'All AB is CD', or $AB(1-CD)=0$. In any case what we have to do when we want to 'express' a hypothetical is to display categorically and unambiguously the actual class-relations between the terms involved, from which that hypothetical can be inferred.

As regards the occasional verbal difficulty of interpretation, the analogy of mathematics will aid us. Take an elementary example. Two persons, each now 50 inches high, have been continuously growing at the rate of 4 and 6 inches, respectively, in the year. What were their heights nine years ago? Again: two persons, each now possessing

£50 of property have been gaining £4 and £6 respectively in the year: what were their properties nine years ago? The obvious numerical answer in each case is 'plus 14', and 'minus 4'. Now if these examples had been treated algebraically, by being thrown into terms of x and y, no one, looking at the problems, could have seen the slightest difference between them. The two would have been stated in identical terms, and the answers would have been the same, viz. $x = +14$, $y = -4$. But whereas the negative in the former case makes nonsense, in the latter it admits of the simple explanation that the man was £4 in debt at the outset. In other words the general algebraical solution contemplates possibilities, in respect of the free range of admissible values, which the material conditions of the case in point may sometimes render absurd.

Now this is just how I regard these complicated hypotheticals. The symbolic expression contemplates possibilities which the material conditions may render irrelevant or absurd. The full generality here (corresponding to the free use of positive and negative values in the above case) consists in the usual freedom of combination, and of equation to zero, of any or all but one, of the results of the class-terms, x, y, z. If the particular example forbids this, it is no fault of the symbolic procedure. Thus, when I say that the two hypotheticals, 'Either there is x, or all y is z', and 'All y is x or z', are formally identical in their implication, being each equivalent to $\bar{x}y\bar{z} = 0$, I take it for granted that y may be combined with either x or z, or $\bar{x}$ or $\bar{z}$; that y itself may $= 0$, and so on.

Compare then the following examples:—Either there are fairies, or all folklore is false:—Either there are householders there, or all the voters are lodgers. Thrown into symbols these are identical, and both yield $\bar{x}y\bar{z} = 0$, which

may be read, 'All y is x or z'. In the latter case this is intelligible enough, implying as it does, that all the voters are either householders or lodgers. In the former case we are made to say that all folklore is either false or fairies. Surely the rational explanation here is that which we adopted before, viz. that the freedom contemplated in the use of our symbols can be exercised in the one case, and not in the other. The example which deals with the voters represents the full freedom contemplated. When however we look at the other example we see that it stands on a very different footing. We obviously take for granted here that there *is* folklore, so that y cannot $=0$: —in fact must $=1$, as a permanent condition. What we really contemplate is nothing more than a pair of alternatives and their contradictories:—fairies (or no fairies), folklore true (or false). It would have been better therefore to have couched the example in the simpler form of 'Either S or P'; viz. 'If not S then P'; or $\bar{S}\bar{P}=0$.

There is another point of some interest which ought to be discussed before quitting this subject. It concerns the legitimacy of a certain extension of the Hypothetical to extreme cases; which, though it forces itself upon our notice in the study of Symbolic Logic, might equally be raised within the bounds of the ordinary system. Logical treatises universally admit that we are at liberty to convert negatively or 'destructively', any hypothetical judgment. Given 'If A is B then C is D' we never hesitate to convert it into the form 'If C is not D then A is not B'. This amounts to putting the second clause on exactly the same footing of general uncertainty as the first, and quite falls in with the current use of the hypothetical. The very form of expression contemplates the non-occurrence of the first event, 'A is B'; and the general application of these propositions

to cases of physical connexion almost amounts to an implication that, though the second clause may happen when the first does not, it cannot be always happening.

The analogue of this in the case of categoricals is, of course, that free right of conversion and contraposition which we discussed in Chap. VI. When, therefore, we put our hypothetical into the form 'P is Q' ($P\bar{Q}=0$), we are at once obliged to reconsider that question. Given two terms P and Q,—each of these being possibly complex,—are we at liberty to equate to zero either of these terms or their contradictories? That is, can a proposition, in which either of these terms stands equal to zero, be admitted as intelligible? Neither popular thought nor Common Logic, as we saw, seems to have made up its mind as between this freedom and the hostile right of general Conversion and Contraposition. Symbolic Logic, on the other hand, fully admits the possibility of equating to zero any combinations of P and Q, subject to the one fundamental condition $PQ+P\bar{Q}+\bar{P}Q+\bar{P}\bar{Q}=1$.

When therefore the hypothetical is identified with the categorical we have in consistency to admit that P may be always, or never, true; and that Q may be always, or never, true. All of these admissions are, of course, decidedly opposed to ordinary conventions. Take (1) the case of P being always true, with the necessary consequence that Q is also always true: 'If the sea is salt then the earth is spheroidal'. All that we mean by the underlying relation is to deny $P\bar{Q}$, viz. the combination of the salt sea and the non-spheroidal earth. Misleading as the verbal rendering may be, we cannot on our explanation call it illegitimate. Take (2) the case of Q being never true, with the consequence that P is never true. 'If the earth is flat then the sea is not salt'. All that we thus categorically deny is

the combination that the earth is flat and the sea is salt[1]. (3) Suppose that Q is always true, with the consequence that P may or may not be true: 'If I have a head-ache then the sea is salt': 'If I have not a head-ache the sea is salt'. This case has been more frequently discussed, and I think that all who have touched upon the question have admitted that, symbolically, such an extension of the hypothetical is the only consistent course. Thus Mr McColl, and Dr Frege (*Begriffsschrift*, p. 3) have recognized that it is quite necessary to divest the proposition 'If A then B' of any suggestion of necessary connexion between A and B; and therefore to extend the range of it so as to cover the case of B being always true, or known to be true independently of A. A paper also by Mr C. S. Peirce (referred to on p. 163) may be usefully consulted upon these limiting cases of hypothetical statements. In the above example we are only supposed to deny that my having a head-ache, or not having it, concurs with the non-saltness of the sea. (4) Suppose that P is never true, and that therefore Q may or may not be true. $P\bar{Q}$ will equal 0, whatever Q may be, and therefore the hypothesis is formally legitimate.

I have called attention to these developments,—perversions, from the ordinary point of view,—because they throw some light upon a discussion in an earlier chapter. It is universally admitted that there is a close analogy between

[1] As *irony*, expressions of this kind are not uncommon:—'Well: if he is an honest man then I am a rogue'.—"She wrote it when the Holy Father wrote the bestiality that posts through Rome put in his mouth by Pasquin":—'when' having much the same force as 'if'. (The Ring and the Book.)

So in the case where the antecedent is regarded as certain. We may meet such an expression as, 'If such enquiries are curious they may still be very useful'; where it is probably meant that they *are* curious, and that their utility is not merely dependent on the curiosity.

the hypothetical and the categorical, in respect of the 'destructive inference' in the case of the former and the contraposition of the latter; and some ordinary logicians (e.g. Spalding: v. p. 152) go so far as to practically identify the two forms, and, to this extent, are in concurrence with the symbolic view. This being so, it seems worth noticing that the attitude of popular thought towards the two forms of expression is not quite the same. In the case of the categorical, we saw that the assumption that the employment of *P* and *Q* in a proposition postulates the existence not only of *P*'s and *Q*'s but also of not-*P*'s and not-*Q*'s, was decidedly questionable. In the case of the hypothetical the corresponding postulate seems quite in accordance with popular usage. In any physical sense of the term, a connexion or uniformity between the occurrence of *P* and of *Q* almost implies that both *P* and *Q* may sometimes fail to occur. Our practical methods of establishing the uniformity of connexion turn largely on the employment of the failures of occurrence,—as witness the familiar Four Methods of Herschel and Mill[1].

The above remarks of course fall in with the general purport of this chapter. My contention is that the really characteristic elements of the Hypothetical,—those which give it the peculiar significance which it possesses in common life and in Inductive Logic,—are of a comparatively non-formal nature. They are to a considerable extent either material or psychological, and therefore the necessity of forcing them into our system will crush much of the life out of them.

[1] To apply the language appropriate to physical connexion to cases in which one of the events is always occurring is,—to employ an apt illustration by Whately,—like speaking of a clock which does not go as being 'right' once in every twelve hours.

Such a loss as this is however inevitable. Whenever we substitute anything resembling machine work for hand or head work, we find that though the former possesses vast superiority of power there are always some delicacies of performance in which it exhibits comparative failure. So it is here. We gain much power in the manipulation of propositions, as we choose to treat them, but it would be vain to pretend that our treatment elicits all the delicate implications which common thought detects in them. These remarks are not directed solely against the Symbolic System: as was shown in a former chapter, when discussing the import of Propositions, the Common Logic also has to disregard many of the finer suggestions and implications which popular thought and speech never fail to recognize. People who lay down a railroad gain in speed and certainty, but they must consent to forego the innumerable hints which are open to those who wander at will amongst the customary devious foot-tracks.

CHAPTER IX.

THE UNIVERSE OF DISCOURSE.

WE have had repeated occasion to refer to the Logical Universe of Discourse in the foregoing chapters, but the present opportunity may be taken in order to complete what is necessary to say upon this topic. As in other parts of our subject, there are three main points which demand enquiry; for in trying to rearrange our subject-matter in accordance with the principles of the Symbolic Logic, we cannot afford to pass over either the conclusions of unassisted common sense, or the rules and assumptions of the ordinary logician.

As regards then the popular way of thinking, the question appears to be this. When we make use of names and resort to reasonings, what tacit limits of reference or applicability do we make? What is the range of subject-matter about which we consider ourselves to be speaking at the time? It is quite certain that every term is not always understood in its full denotation, for nothing is commoner than to meet an objection, or supposed exception, with the retort: Yes, but I did not intend to refer to *that* when I used the words: I was confining myself to so and so.

The tacit limits thus imposed upon the range of applicability of our terms are partly of a very general character:

partly special or individual. As regards the former, it is clear that, from this point of view, we regard positive and negative terms as standing on a very different footing. The word 'black', for instance, is used approximately in its full range of applicability, whereas many restrictions are put upon that of 'not-black'. We might, if we pleased, define it as comprising everything which is not black:—including, say, the Geological Glacial Period, the claims of the Papacy, the last letter of Clarissa Harlowe, and the wishes of our remote posterity. But the reference is so plainly to *colour* that no reasonable person could have any hesitation in the usage. And similarly in most other cases.

True negative names of the 'not-*X*' type are not very frequent in popular speech, being mainly an invention of the logician. Still they do sometimes occur as subjects or predicates of propositions, though usage prefers that the negative should be introduced into a sentence rather than be simply joined to a term by a hyphen:—e.g. 'What is not conceivable is no fit subject for instruction'. This is the usual resource when the class to be indicated happens to be narrower, or more conveniently assigned in this way than by the use of positive terms. Of course if we class amongst negative terms such words as 'inhuman', 'unnatural', and so forth, it becomes clearer still that they are strictly limited to a portion only of the whole range to which they could formally extend by right of mere negation.

The same general question is sometimes raised in another form by the enquiry whether we have any pairs of terms in our language which are strict contradictories: understanding by this that they shall be not only mutually exclusive but collectively exhaustive. That we have plenty which in mere form are contradictory, is plain enough:—

almost every one of the negative terms alluded to above represents one element of such a pair. But ordinary language being determined by relation to human wants, we shall almost invariably find that such terms are restricted in their practical application to some well understood universe. A large number of the contradictories in common use stand to each other in a material rather than a formal relation, and the two together cover a very small portion of the totality of existence. Thus 'British' and 'alien' are equivalent to British and not-British, provided we understand that we are talking only of human beings. When we employ an artificial term like not-British, which does not possess the same clearly understood limitation as 'alien', we are obliged tacitly to apply the limitation for ourselves, and then the conception of the universe has to be introduced.

In the foregoing remarks we have considered such restrictions on the full generality of our terms as are almost universally recognized. But besides this every speaker has a more or less personal and individual restriction to impose. He uses general terms, but he uses them with a clear understanding that he then and there intends them to apply to a portion only of the objects which they strictly comprise. He says, 'All X is Y'; but the tacit proviso is to be supplied, 'I mean only to refer to the X's which are Z'.

This conception of the Universe, therefore, only presents itself in popular speech as a restriction,—either general or personal,—upon the full range of potential application of the terms employed. We must now notice briefly what the ordinary logician has to say upon the matter. The use of the word 'Universe' was first familiarized by De Morgan[1],

[1] *Camb. Phil. Trans.* VIII. 580. He speaks there of "inventing a new technical term".

but the conception is one that is suggested at more than one point in the traditional treatment:—everywhere, that is, where symbols or terms which are potentially general in their application are limited either by general convention or by individual intention[1]. Hamilton's doctrine of the Quantification of the Predicate is perhaps a case in point. We say, generally, All X is Y; but it is maintained, by the advocates of this doctrine, that we may *mean* that X is some only of the Y, and that, if so, we should express the limitation in words. The old doctrines of a *summum genus* and of the Categories raise the same questions. As these latter are now often understood (e.g. by Mill) it is maintained that what Aristotle and his followers were aiming at was a classification of all conceivable entities. But this view was expressly repudiated by most of the old logicians. They regarded each category as a sort of universe by itself, higher than which we need not go, and they distinctly maintained that all these categories together did not embrace the sum-total even of things existent[2]. Another department in which the same conception seems to make itself prominent is in the discussions about the nature of 'infinite' or indefinite terms and propositions. The subject is too intricate for discussion here. I will therefore merely refer to the form in which this doctrine was held by a very eminent thinker who was but little restrained by traditions of the past. Students of Kant will remember the three-fold division

[1] Perhaps the nearest approach to a regular discussion of this subject, in the old treatises, is to be found under the head of *Suppositio*: see, for instance, Sanderson.

[2] There were familiar memorial lines enumerating the *transcendentia* which lay outside the categories. The version given by Seton is,

Vox artis, consignificans, privatio, fictum,
Pars, Deus, excedens, complexum, vel polysemus,
Ista categoriis haud possunt verba locari.

of propositions which he makes, in respect of their quality, into positive, negative, and infinite. Verbally, of course, it is easy enough to say that we must either assert that A is B, or deny that it is B, or (couching the latter in affirmative form) assert that A is not-B; and we may readily admit that there is some conventional difference of signification between these various cases. But is there any difference whatever, of which logic should take account, between the last two? On any rigid class view of the nature of predicates it is impossible to extract more than two divisions; for, that to exclude a thing from a boundary is to include it somewhere outside that boundary, that to deny that any thing has a given attribute is to assert that it has not that attribute, seems indubitably clear. I suppose that the idea underlying the distinction is this. When we deny that A is B we think of A as a whole, and B as an attribute and therefore as a whole, so that the judgment is finite in both terms. But when we say that A is not-B and try to consider this not-B as an attribute, we have forced upon our notice the vague amplitude of its extent; and therefore, when we do not make appeal to a limited universe, we must recognize that the judgment is in respect of its predicate an infinite or indefinite one.

Whether I am right or wrong in this last remark it will equally serve to call attention to the view which the Symbolic logician is bound to adopt. Taking, as we do, a strict class view of the nature of propositions we meet the difficulty by flatly denying that the class not-X need be more 'infinite', or in any way more extensive even, than X. The notion that this is so is simply a survival from the traditions of common speech, and is one of which the symbolist should rid himself as speedily as possible. Not-X is of course always the contradictory of X, but there is no reason to suppose that

the former symbol is more appropriately applied to classes which are essentially negative or are popularly regarded as such[1]. There may be practical reasons of convenience for thus assigning our symbols, but as far as any reasons of principle are concerned we might interchange X and not-X all through our logical processes without the slightest change of symbolic significance. There is nothing to hinder us from putting not-X to stand for the few and highly specialized members of some narrow class, and X for the innumerable and heterogeneous individuals which do not belong to it.

When thus regarded, the conception of a universe is seen to be strictly speaking extra-logical; it is entirely a question of the *application* of our formulæ, not of their symbolic statement. It is quite true that we always do recognize a limit, sometimes express but more often tacit, as to the extent over which not-X is to be allowed to range; and also that we not unfrequently do so in respect of X itself, so long as these expressions are set before us in words and not in symbols only. Between them, X and not-X must fill up the whole field of our logical enquiry; they can leave nothing unaccounted for there. But when the question is asked, How wide is that field? the only answer that can be given is, Just as wide as we choose in any case to make it. Whether the practical imposition of these limits does most to curtail

[1] I cannot therefore agree with Prof. Wundt when he says (*Logik* I., 233) that 'Boole's view rests upon the wide spread logical error according to which the concept non-A is referred to the infinitude of the concept world'. (It seems to me that Prof. Wundt's treatment of his subject is in several places somewhat marred by his not having shaken off the language and tone of conceptualism.)

It deserves notice that one of the earliest writers to apply symbolic notation to Logic,—Segner,—has called attention to this indifferent symbolic application of X and not-X. Employing ($-$) to mark the contradictory, he says that if we like to put X for *non-triangulum* then $-X$ will stand for *triangulum*. (*Specimen Logicæ*, p. 71.)

the range of X or of not-X is of no significance, for this will depend upon the arbitrary assignment of our symbols in the stating of our premises.

This limitation of our universe, tacit or avowed, may take a variety of forms. In this respect Boole's view has, I think, been misapprehended by some writers (as by Dr Macfarlane)[1]. He seems to think that Boole just drew, as it were, a definite outline to mark the limits, and then considered himself bound to take every kind of logical entity to be found within those bounds. I cannot perceive that that was his view, and should certainly reject any such interpretation myself. The limits of application of our formulæ seem to me in every respect open to our own choice. They may take the form of any order or plane of existences, as well as that of any boundary line on such a plane. For instance, we are applying, say, the terms European and not-European. We may extend our universe so as to embrace the sum-total of logical existence, in which case European includes things other than men, and not-European includes the unlimited myriads of entities which people that heterogeneous domain. Or we may restrict it to *man*, in which case not-European is limited to men of other quarters of the world. Or it is equally open to us, in the case of any special example, to confine it to the present British House of Commons, in which case European is limited to some 669 persons, and not-European is represented I believe, by one.

'All' and 'nothing' therefore, in any application of our formulæ, are to be interpreted in accordance with the material

[1] "It appears that what Boole means by the universe of discourse is not the objects denoted by a Universal Substantive, but a definite part of the whole realm of things,—a limited portion of the physical universe, with all the entities which are or can be imagined to be in it, whether mental or physical, ponderable or imponderable, atomic or complex" (*Algebra of Logic*, p. 6).

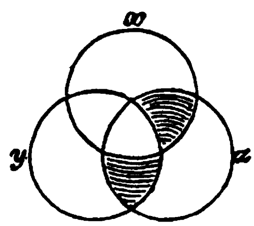
or actual limits which we may decide to lay down at the outset of the particular logical processes in question. The *all* of some reference may, as it happens, be absolutely all, in the sense that the widest extension of our universe would not yield any more of it[1]. Thus no extension of the universe beyond *man*, would yield any other specimens of 'rational animal' than those which are human. And the *nothing* of some reference may merely mean nothing *there*, whereas the term may apply to any number elsewhere, possibly to 'all' else that exists. All applications of our logic are, as remarked, at our free choice; we might limit our application of the terms 'good' and 'not-good' to the London cabs with odd numbers, and every logical rule will hold valid as well as if we had selected a less absurd sort of universe. Refer, for instance, to example (10) in chapter XIII. We are there concerned with three girls, *A*, *B*, and *C*, and with one fact only about them, viz. whether they are in or out. We need take account of nothing but this for the purpose of our problem. Although therefore we use general symbols, eight possibilities exhaust our entire reference. These eight possibilities constitute our universe in this case.

It has been said above that this question of the Universe only arises when we *apply* our formulæ. Now diagrams are strictly speaking a form of application, and therefore such considerations at once meet us when we come to make use of diagrams. I draw a circle to represent *X*; then what is outside of that circle represents not-*X*, but the limits of that outside are whatever I choose to consider them. They

[1] As Prof. Peirce puts it paradoxically, 'In a really unlimited universe every universal proposition, not tautologous, is false; and every particular proposition, not absurd is true'. This follows, of course, from our conception of the respective characters of these two kinds of propositions. The former put an actual limit upon what is conceivable: the latter establish the existence of something conceivable. (*Studies*, p. 182.)

may cover the whole sheet of paper, or they may be contracted definitely by drawing another circle to stand as the limit of the Universe; or, better still, we may merely say vaguely that the limits of the Universe are somewhere outside the figure but that there is not the slightest ground of principle or convenience to induce us to indicate them.

The settlement of the Universe being therefore purely a question of application it can never be indicated by our symbols, for these must in themselves be perfectly general. They know nothing of any kind of limit except what is purely formal. When it is asked, What are the limits of not-x? the symbolic answer is invariably the same, 'all that is excluded from x is included in not-x'. It is only when we go on to enquire what is meant by 'all' that the question of a limit comes in, and this is a practical matter involving the interpretation of our data. Hence, for instance, we ought not to say that in the expression $zx = zy$, we necessarily mean that "in the universe z, all x is the same as all y". That we may make z our universe in this case, as in any other, is indisputable, and, if we do so, then the above is the true explanation of the statement; but there is nothing in the statement to compel us to make it there, or to do more than suggest to us to do so. On the contrary, the employment of any symbol z immediately intimates not-z, and unless we are told (on material grounds), or decide for ourselves arbitrarily, that there shall be no not-z, we should naturally infer that the universe will find place for something of that sort. The statement $zx = zy$ would, on a diagram, be thus represented;—



(for we must simply regard 'zx which is not y' and 'zy which is not x', as being abolished). It is clear that within the universe of z, x and y coincide; but then so they may also to some extent outside it, to say nothing of there being also a place provided outside for what is x alone, and y alone, and neither of the two. Similarly we can see that within the universe x, z is but a portion of y; and that within the universe y, z is but a portion of x[1].

I cannot therefore agree with Mr Macfarlane (*Algebra of Logic*, p. 29) that "Every general proposition refers to a definite universe, which is the subject of the judgment... For example, 'All men are mortal' refers to the universe 'men'. 'No men are perfect', refers to the same universe". I prefer to say that there is no indication here of what the universe may be; this being a matter of private interpretation or application, in no way suggested by our symbols. Moreover, on the symbolic system it is universally admitted that the distinction between subject and predicate is lost: why then are we to consider that 'no x is y' has x as its universe rather than y?

The outside of our Universe itself is of course simply disregarded. It must not only not exist in the sense of being affected by negative attributes only, but in that of having no attributes whatever, positive or negative. It contains no compartments even, which we can speak of as either empty or occupied. We simply do not suffer our minds to dwell upon it. The outside of any particular universe may in fact be considered to stand in much the same relation to all

[1] This may be exhibited symbolically thus:—Put $z=1$, then $x=y$. That is, when z is made 'all', x and y are co-extensive. Similarly put $x=1$, and we have $z=zy$, or 'All z is y'. It is clear that in making these changes we have interfered with the statement as originally given to us. That statement in no way called for these restrictions, though it lays itself open to them.

possible logical predication that the field of 'view' at the *back* of our heads stands to all possible colours. The fact that we thus regard the material extent of that sum-total of things which in any given instance makes up our Universe as a matter of application, rather than of pure theory, is really a reason for representing the Universe by the symbol for *unity*[1]. The standard formula $xy + x\bar{y} + \bar{x}y + \bar{x}\bar{y} = 1$ declares that the four classes on the left combine to make up 'all'. This is necessarily true, and from the formal point of view nothing more can be said. If it be asked what this 'all' may comprise in any particular case, we can only reply that the decision rests with the logician himself. It is to all intents and purposes a part of his data, for though it may not be stated in the premises he must be supposed to bring a knowledge of it to his solution of the problem. A perfectly general formal symbol, such as 1, lends itself well to this ever-varying individual application.

[1] The main grounds for choosing *unity* for this purpose were given in Chap. II. They were connected with the process of 'multiplication' adopted in our logical system. As regards the process of 'addition' a certain awkwardness must be admitted on the non-exclusive plan (it does not appear when, with Boole, we make our alternatives exclusive) whatever symbol we employ. For instance R. Grassmann (*Begriffslehre*) makes use of the letter T. Adopting as he does the non-exclusive plan he writes $a + T = T$, in order to express the fact that the Universe, since it includes a, cannot be increased by any addition of a to it. Perhaps less violence is done to acquired associations by writing $a + T = T$ than by writing $a + 1 = 1$. The companion formula however, $a \times T = a$, is less convenient than $a \times 1 = a$. Those who adopt the symbol ∞ for the Universe (Wundt, *Logik*, p. 233, Prof. C. S. Peirce, *Am. Ac. of Arts and Sciences*, 1870; *Am. Journal of Mathematics*, Vol. III.), adopt in consistency, on the same non-exclusive scheme, the results $a \times \infty = a$, $a + \infty = \infty$.

CHAPTER X.

ON DEVELOPMENT OR EXPANSION.

THE process called sometimes Development, and sometimes Expansion, is the most fundamental and important with which we shall have to concern ourselves. In fact when the nature and results of this process are fully understood the main task of Symbolic Logic is grasped. Both of these terms (Development and Expansion[1]) possess some symbolic propriety, as will be seen when we come to give rules for the performance of the operation in question; but on merely logical grounds such an expression as 'continued dichotomization', or 'subdivision', would seem to be more appropriate.

Every one who has read a treatise on Logic is familiar with the fact that any assignable class admits of dichotomy, or division into two parts, x and not-x respectively, whatever quality x may stand for. One or other of these two parts may of course fail to be actually represented, but both

[1] The objection to these two terms is that they do not readily offer any appropriate *correlatives;* whereas when we regard the operations themselves, we see that the splitting up of a class into its ultimate elements, and the recombination of these elements into a single group, are relatively inverse operations. 'Subdivision' and 'Aggregation', though not without objection, seem to satisfy this condition of standing as convenient correlatives.

cannot thus fail; in other words, they may be regarded, as already remarked, as compartments into one or other of which every individual must fall, and into one or both of which every class must be distributable. At this point common Logic mostly stops in practice. It is clear however that we have thus made but a single step along a path where indefinite progress is possible. Each of the classes or compartments thus produced equally admits of subdivision in respect of y, whatever y may be; and so on without limit. The subdivision in respect of one class term (x) gives two classes, x and not-x; that in respect of two terms gives four classes; with three we obtain eight, and so on. With n terms thus to combine and subdivide we have a complete list of 2^n ultimate classes.

This dichotomous scheme is of course absolutely complete so far as it extends. It contains the provision or the raw materials for the statement of every purely logical proposition which can possibly be framed by employment of the terms in question. It may therefore be regarded as a sort of framework for all possible propositions involved in, or expressible by, the given terms. If, for instance, we have two terms x and y, then the four sub-classes indicated by xy, $x\bar{y}$, $\bar{x}y$, and $\bar{x}\bar{y}$, comprise all the elements which can possibly be needed or employed for the purpose of constructing propositions out of the terms x and y. This was explained in a preceding chapter where we drew up, on this suggestion, a scheme of elementary forms of propositions. To these we might have added many more by combining the elements two and two, or three and three together.

Of course this dichotomous subdivision of terms is not Logic, but rather a substructure of Logic. It is an orderly process for assigning all the elements which we can need in our reasonings, and the performance of it is therefore

rather a preliminary to reasoning than reasoning itself. It is however an absolutely necessary preliminary when we propose to occupy ourselves with complicated propositions or groups of propositions, and we shall accordingly proceed to discuss it.

The process of subdivision thus indicated is, as a rule, easy enough to carry out by a mere inspection, unless we are concerned with complicated class expressions. Suppose for instance,—to begin with a very simple example,—that we had the class group $xy + \bar{x}z$ to deal with in this way. It is clear at once that the term xy admits of no further subdivision of this kind, as regards either x or y; for it can yield no term of the not-x or not-y description. Accordingly the only subdivision open to it is in respect of z, so we divide it into the parts which are, and which are not z, viz. into xyz, and $xy\bar{z}$. Similarly $\bar{x}z$ admits only of subdivision in respect of y, and yields the terms $\bar{x}yz$ and $\bar{x}\bar{y}z$. And here the process stops, for we are only supposed to have the three terms x, y, and z in hand. Indeed they are the only terms entering into the given expression. Accordingly the result of the process is given by $xyz + xy\bar{z} + \bar{x}yz + \bar{x}\bar{y}z$.

So much of what has to be said on the subject of this Development has already been anticipated in previous chapters, that it does not seem that much more explanation is called for. We shall have to consider the subject, in its wider symbolic aspect, when it assumes a form which is very far from being familiar and obvious; but at present I prefer to look at it in the light of ordinary logical considerations. Such simple remarks as remain to be said under this head may be gathered up as follows:—

I. It will be observed that in the above example we only developed the expression $xy + \bar{x}z$ in respect of the class terms actually involved in it, viz. x, y, and z. There was

however no necessity for thus confining ourselves; every subdivision which we reached might have been still further subdivided, if we wished, in respect of a, b, c, or any other succession of letters regarded as representing class terms. The reasons for not doing this are however obvious. Any degree of subdivision is useful which turns upon class terms about which definite suggestions are given to us, that is, which enter into the data of the problem before us. But to proceed beyond this point, by introducing terms about which no suggestions are given to us, can be nothing but an idle exercise of our rules of operation.

Of course each component element of the expression which is given to us to develop will be treated in respect of terms which *it* does not contain; otherwise indeed there could be no development. Thus xy did not involve z, nor did $\bar{x}z$ involve y. The rule is that every element must be developed in respect of every term which we may have to take into account, that is, which the whole expression before us involves. Hence follows one very important consideration. We shall often find *unity* entering amongst our symbols; and this, being a class term, must admit of development like any other. No terms visibly occur in it, but representing as it does the sum-total of all things, all conceivable classes must be contained in it. Hence unity must be developed, in any given case, in respect of *all* the class terms involved in the expression in which it occurs. Take for instance, $1 - xy - \bar{x}\bar{y}$. Here two terms (x and y) occur in combination. When we develop 1 in respect of these two, we obtain $xy + x\bar{y} + \bar{x}y + \bar{x}\bar{y}$. The other two terms in the given expression are already in their ultimate state of subdivision. Hence the whole expression when developed resolves itself into $x\bar{y} + \bar{x}y$. As it happens here, this is actually a simpler and shorter expression than it was before development. As

a rule the reverse is the case, this process of subdivision generally multiplying the number of final terms; to such an extent indeed when the terms are numerous as to make the statement of the result very complicated.

II. Every class other than unity, howsoever composed, must be logically *less* than unity. This follows from the very meaning of our symbols; for 1 stands for 'all things', and x stands for a limited selection from all. Hence no class term, or group which can constitute a class, will contain *all* the elements which have to be referred to in the process of developing it. Some will contain more of these ultimate elements, others will contain less, but the development of unity only can contain them all. Some indeed may already happen to be given in their lowest terms and so admit of no further development. Thus if we took xyz and developed it with respect to the three symbols which it contains, we should merely get the same result over again, every other combination vanishing from the development.

III. If the original class group which was given to us to be developed contain none but mutually exclusive terms, no element in the development will appear more than once in our result. If otherwise the fact will appear immediately in the development, by one or more of the elements having some other factor than unity;—2, 3, or so on. Take for instance $x + y$ and develop it; it will yield the result $x\bar{y} + \bar{x}y + 2xy$. This of course reintroduces the question of overlapping classes, and the proper form of expression of an alternative. Those who hold, with Boole, that mutual exclusiveness is the rule, will regard this plan as a means of indicating the extent to which the original expression had been faultily expressed: others will consider it as an indication of economy in expression. Both will agree that when the question is whether or not a certain group of propositions

involves any surplusage of statement this is a very ready mode of directing attention to the nature and amount of this surplusage. Examples in point will be given in a future chapter.

Up to this point in our investigation of the process of Development, we have resorted to no other considerations than such as are suggested by ordinary Logic, and are perfectly explicable within its province[1]. Since everything must be either x or not-x it stood to reason that every class which did not expressly by its form belong to one or other of these two divisions alone, must be capable of being split up into two parts, one of which belongs to each. And this process, once commenced with some one class term as the dividing element, could be continued with another, and so on indefinitely, or until all the class terms assigned in the data were exhausted.

What we now propose to do is not to trust to mere acuteness to carry out this process, but to ascertain whether we cannot give a perfectly general symbolic rule of operation for effecting it. This was one of the great steps of generalization introduced by Boole. Take then for examination a group of class terms a trifle less simple than those given above, for example $x+y+xyz$, and develop this with respect to x. The first and third of these terms remain unchanged, for since they involve x they cannot yield any not-x part. The second term splits up into xy and $\bar{x}y$. The whole expression thus becomes $x+y(x+\bar{x})+xyz$. The rule of formation is readily seen. Every term in the given expression which involves either x or $\bar{x}$ is left as it stands, and

[1] It deserves notice that Lambert had got as far as this in Logic: e.g. "Man drücke die eigenen Merkmale des a durch $a|b$ aus, und die eigenen Merkmale des b durch $b|a$; so hat man $a|b + b|a + ab + ab = a + b$" (*Log. Abhandlungen*, I. 11).

every term which does not involve x is multiplied by the factors x and $\bar{x}$. Of course, in a sense, all the terms are left unaltered in value; for subdivision, when all the elements are retained, leaves the aggregate undisturbed. But then to resolve the total class into its elementary parts and to retain all these parts before us, is the very process which we are proposing to carry out.

As thus deduced, the Development took the form $x + y(x + \bar{x}) + xyz$. Now this may be differently arranged, if we pick out separately and group together the x and not-x parts, for it then becomes

$$(1 + y + yz)\,x + y\bar{x}.$$

When put into this form we see that the following rule of operation for obtaining the development may be laid down. 'Write 1 for x all through the given expression, and multiply the result so obtained by x: then write 0 for x all through it, and multiply this result by $\bar{x}$. The sum of these two results is the full development of the given expression with respect to x'.

This rule of operation lends itself to a remarkable symbolic expression; for the result of it is conveyed at once by the statement, 'the development of $f(x)$ is $f(1)\,x + f(0)\bar{x}$'. This is the rule given by Boole almost at the commencement of his work, and which plays so large a part in his method[1]. As regards the logical interpretation of it we shall have more to say hereafter, but at present I will merely give a word or two of explanation of its symbolic construction as it thus stands. It need not be added, after what has been already said, that $f(x)$ is to be regarded as being

[1] Boole gave only formal proofs of this rule. Thus, assume $f(x) = ax + b\bar{x}$. Putting $x = 1$ and 0 respectively, we have $a = f(1)$, $b = f(0)$; hence $f(x) = f(1)\,x + f(0)\,\bar{x}$. He then applies the formula at once to the development of a fractional expression. (*Laws of Thought*, p. 72.)

merely a perfectly general symbol for any class, or group or arrangement of classes, which involves x in it. The symbol $f(1)$ stands, as usually in mathematics, for this same class group when it is altered by changing every x into 1, and leaving every other letter unaltered; whilst $f(0)$ stands for the same class group when x is changed into 0 and no other change made. Hence the expression $f(1)x+f(0)\bar{x}$ must be regarded as being merely a symbolic representation of the rule of operation given above.

In the above investigation we were only supposed to subdivide the given class expression in respect of *one* of the terms involved in it, viz. x. It is equally open to us however to proceed further in the same course of subdivision. Presumably the given expression was built up of a number of class terms aggregated together, such as x, y, z, &c.; of which terms we have only taken one into account. Now it must be remembered that the result of this first operation will be of precisely the same general character as the original expression with which we started. We were supposed to start with a class expression, indicated by $f(x)$; and we get from it another class expression, indicated by $f(1)x+f(0)\bar{x}$. The only difference between them is that the latter is more subdivided than the former, having been broken up in respect of one of its component elements, viz. x.

We want then to develop $f(1)x+f(0)\bar{x}$ in respect of, say, y. We might proceed by the process which we first started with, but it will be simpler to adopt the symbolic process which was seen to yield the same result. The rule was to write 1 for y all through and multiply by y, and then write 0 for y all through and multiply by $\bar{y}$, and add these together. This rule applied to $f(1)$ gives us $f(1, 1)y+f(1, 0)\bar{y}$; where $f(1, 1)$ stands for what $f(x)$ becomes when y as well as x is put equal to 1, and $f(1,0)$ stands

for what $f(x)$ becomes when x is put$=1$, and $y=0$. Similarly $f(0)$ develops into $f(0, 1)\,y + f(0, 0)\,\bar{y}$, where $f(0, 1)$ is $f(x)$ with $x=0$, and $y=1$, and $f(0, 0)$ is the same expression with both x and $y=0$. Hence the whole result thus obtained is

$$f(x, y) = f(1, 1)\,xy + f(1, 0)\,x\bar{y} + f(0, 1)\,\bar{x}y + f(0, 0)\,\bar{x}\bar{y}.$$

Of course we need not stop here. If there was a third class term z in the original expression, this must appear in one or more of the factors $f(1, 1)$, $f(1, 0)$, &c. Hence the process of development or subdivision may be repeated a third time, yielding 8 ultimate elements; and so on, until we have taken every class term in the original expression into account.

In this expression, as will be more fully pointed out presently, the factors $f(1, 1)$, $f(1, 0)$, &c. are not logical class terms involving x, y, &c., at least not when the Development is fully completed; for by that time every x and y has been changed into 1 or 0, and has accordingly disappeared from the result. Hence these expressions are, symbolically, purely *numerical*[1] quantities; whilst logically, they are, as will be shown, directions to take into account the whole, an uncertain part, or none of the class to which they are prefixed as factors. Thus, in this case, xy denotes a class; $f(1, 1)$ tells us whether or not that class is to be included; and so on all through.

The point therefore at which we have at present arrived is this. By simple and intelligible logical steps we can break up a composite class into all the elementary classes of which it consists; and we can show that identically the same result can be obtained by the symbolic rule of operation, 'for $f(x)$ write $f(1)\,x + f(0)\,\bar{x}$'.

[1] In the restricted sense of 'numerical' admitted in our Logic: see further on for explanation.

We must now see under what sort of restrictions such a rule as this is obtained, and whether any of these restrictions are removable.

To begin with; is it necessary that the class group which we thus develop should be stated as an aggregate of mutually exclusive terms? By no means; in fact one use of Development is to detect whether or not the component elements do thus overlap each other. Treat $x + y$ in this way and it becomes $2xy + x\bar{y} + \bar{x}y$, thus reminding us that the common part is xy.

This will indeed be obvious if we remember that Development is nothing but subdivision into its ultimate class elements, and that all that is meant by mutual exclusiveness is the avoidance of the double counting of the overlapping part; this being so, every doubly counted element in the development must necessarily stand out in relief from amongst the others by being multiplied by the factor 2. Similarly if it is counted thrice over. Thus develop $xy + xz + yz$ and we have $xy\bar{z} + x\bar{y}z + \bar{x}yz + 3xyz$, thus reminding us that each of these three terms involves one part peculiar to itself, and one part (xyz) common to all three.

There is however a far more important extension than this to be considered in the application of our formula. Have we a right to apply it to *uninterpretable* expressions, either at first hand or in the process of passing through such expressions in case we are led to them? There is no doubt that Boole held this opinion himself. He regarded this rule of Development, apparently, as a sort of engine potent enough to reduce to a series of intelligible logical terms expressions which as given to us had not a vestige of intelligible meaning in them. He appeals, in justification, to the practice of the mathematicians; especially to their

employment of the symbol $\sqrt{-1}$, as offering an analogy and justification for this corresponding step in Logic.

If we were forced to adopt this view a very difficult enquiry would have to be entered on. Fortunately however it does not appear that any discussion of this kind is really called for, at least not on the principles adopted in this work. It is to *fractional* forms only, such as $\frac{x}{y}$, that it could ever be proposed to apply our formulæ in any supposed uninterpretable application of this kind,—for it would be a mere sportive misapplication of them to think of making them treat such expressions as $\sqrt{x}$, $\log x$, and so forth,—and to fractional forms we have seen that a very easy logical explanation could be given.

The general justification and explanation therefore of the application of our rule $f(x) = f(1)\,x + f(0)\bar{x}$ to expressions of which $\frac{x}{y}$ is the simplest type, ought not to offer any difficulty. We have already explained these expressions, in detail, by purely logical considerations, so that all that remains at present is to compare the results thus obtained with those deducible from the formula for Development. The reader will remember that we discussed two slightly different ways of approaching the form $\frac{x}{y}$, but that in each case alike that form stood for a *logical class*. It only differed in fact from such a form as xy, in respect that the desired class was more remotely indicated by an operation instead of being directly set before us as a result; and that, the operation by which it was to be derived being an inverse one, it was consequently indefinite in some respects as to its limits. But none the less was it a logical class. On my view therefore the distinction in question is by no means

one between the interpretable and the uninterpretable. It is merely one which may be described *symbolically* as that between integral and fractional forms, and *logically* as that between direct and inverse processes. But this distinction, though by no means a profound one in principle, is sufficiently important in its results to deserve separate consideration here.

We shall therefore apply our formula to $\frac{x}{y}$ with no more hesitation than, for example, to $x + \bar{x}y$. When we do so, developing it in accordance with the rule

$$f(x, y) = f(1, 1)\, xy + f(1, 0)\, x\bar{y} + f(0, 1)\, \bar{x}y + f(0, 0)\, \bar{x}\bar{y},$$

we obtain

$$\frac{x}{y} = \tfrac{1}{1}\, xy + \tfrac{1}{0}\, x\bar{y} + \tfrac{0}{1}\, \bar{x}y + \tfrac{0}{0}\, \bar{x}\bar{y} \ldots\ldots\ldots\ldots\ldots (1)$$

The result obtained by purely logical considerations in the third chapter, it will be remembered was

$$\frac{x}{y} = xy + v.\bar{x}\bar{y}, \text{ with the attendant condition } x = xy \quad (2).$$

A comparison will show the complete identity of these two results. The first term is the same in each, for $\frac{1}{1}$ is of course the same as 1, the logical sense of which has been already settled. So is the third, the only difference being that we had before simply omitted it as not occurring, whereas here it is expressly noticed and rejected. The fourth term again is identical in each case, or may fairly be considered so. Our factor v stood for 'a perfectly uncertain portion, some, all or none': it was proposed in despair of finding any suitable logical word which should possess the same degree of indefiniteness, for of course *some*, in its recognized significations, would not answer our purpose. Now this is exactly the well-known meaning of $\frac{0}{0}$ in mathematics,

which indicates perfect indefiniteness[1]. So much is this the case that but for the wish not to excite prejudice we might have appealed to it at once instead of using the symbol v, but for the future we shall feel perfectly at liberty to resort to it[2]. There still remains one term, the second, which the formula gives in the peculiar shape $\frac{1}{0}\,x\bar{y}$. This term we had omitted, though not on the ground on which $\bar{x}y$ was omitted, —viz. that we were not to include it in $\frac{x}{y}$,—but on the ground that *there was no such class.* Whichever way we

[1] In each case the indefiniteness extends over the whole admitted range. In mathematics that range extends from 0 to ∞, with us from 0 to 1; so that there is this degree of difference of application in the two cases.

[2] It may be remarked that there is an anticipation of this logical fractional form by H. Grassmann, when he expresses the result of a generalized process of division in the form $C+\frac{0}{B}$, C being a particular value of the quotient and $\frac{0}{B}$ a term which will ordinarily vanish but must be admitted as a symbol. He does not however apply it to Logic, nor work it out in detail, but his explanation is worth giving:—'If $B\,.\,C=A$, we have the quotient in the form $\frac{A}{B}$. Now any value which substituted for C satisfies this equation may be regarded as a particular value of this quotient. Every such value will admit of being produced by addition from the value C, and indeed the portion to be added to C when multiplied by B must give *zero* if the product is to remain equal to A, and any such added portion will leave the product equal to A. Now we may represent generally such a portion as when multiplied by B will yield 0, by $\frac{0}{B}$; and we may say that if C is a particular value of the quotient, and B the divisor, then the complete value of the quotient will be $C+\frac{0}{B}$,' (*Ausdehnungslehre*, p. 213). The fact that he generalizes the process indicated by division, by admitting a surplus term (an indefinite one, observe), which will vanish on 'multiplying' again by the divisor, is clear, and this constitutes a striking analogy to the form $\frac{x}{y}=x+\frac{0}{0}\bar{y}$. It must be remembered that this latter also might be written $x+\frac{0}{y}$, since $\frac{0}{0}\bar{y}$ is really the same as $\frac{0}{y}$.

express it, whether in the shape $x = xy$, or $x\bar{y} = 0$, we saw that the condition that there is no $x\bar{y}$ in existence was necessarily presupposed in the mere proposal to accept and interpret the expression $\frac{x}{y}$. Now the meaning of $\frac{1}{0}$ in mathematics is *infinity*. What then is meant by offering us, in a simple class expression, a term multiplied by infinity? Surely that there is no such class in existence, for this is the only way of escaping the consequent absurdity. On this view then the two expressions above given agree in all their details, as we might expect that they would agree.

In speaking thus, however, we must not be understood to be merely borrowing from mathematics. It has been already shown (p. 175), that $\frac{0}{0}$ has a strict logical signification, representing the class of which if we take 'no part' we obtain 'nothing', and is therefore perfectly indefinite. Similarly with the other factorial expressions just introduced. Thus $\frac{1}{1}$ stands for a class such that 'all' of it = all that is, and is therefore 'all' itself, viz. 1. So $\frac{0}{1}$ stands for the class 'all' of which = 0, and must therefore = 0. And $\frac{1}{0}$ attempts symbolically to represent a class such that when we take 'none' of it we yet obtain 'all', and thus represents an impossibility.

It may be remarked that the same inference as to the non-existence of the affected class would have to be drawn if any other definite factor than 1 or 0 had presented itself. For a class rigidly expressed, that is, only once counted, cannot have as one of its constituent elements a portion which is twice or oftener reckoned, since this would really be to compare disparate things. So, if we met the expression $z = x + 2\bar{x}y$, where z was a simple class term, it would necessarily require that $\bar{x}y = 0$.

So far then it is clear that the application of our general

symbolic formula to such expressions as $\frac{x}{y}$ yields nothing more than could have been secured by familiar logical considerations. Why then resort to it at all? For two reasons. In the first place though we *can* get at the full result without such help, it is improbable that we always *should* do so. Without some rule of procedure which should direct our attention to every possible class in turn, and pronounce not merely upon its inclusion and exclusion, but also on its rights of existence under the data assigned, it is hardly possible but that some should be omitted. This, of course, might be secured by any method even of the most tentative kind which took the whole table of possible combinations of the terms as its guide, and insisted on considering them each in turn in relation to the given premises. But there seems to me a far more important reason than this for advocating the use of formulæ such as that in question. As this was insisted on in the Introduction, I will merely call attention to it again in a few words here. The speculative advantages to be gained by really comprehensive logical theorems far outweigh any mechanical saving of trouble which they may secure. To understand the nature of an inverse operation as such; to generalize as far as possible familiar processes; to acquire an intelligent control of symbolic language, as distinguished from a mere mechanical facility in using it, —which can only be done by constantly interpreting its results, especially in limiting cases, and checking them by comparison with the results of intuitively evident processes, —these and such as these are the great merits of a proper study of Symbolic Logic. It is well worth while to take some trouble in understanding $f(x)$ and the processes performed upon it, in order to secure such advantages as these.

We have now given a complete explanation of this formula in its application to those fractional forms of which $\frac{x}{y}$ may be taken as a sample; that is, to forms in which both the numerator and the denominator are themselves intelligible class terms as they stand. But it must be noticed that we shall often have to apply it, and shall find it applied successfully, to expressions which are not so simple as this. For instance, take the expression $\frac{z-x}{\bar{x}}$. The numerator of this, or $z-x$, is not a properly stated class term, for we cannot take x from z unless we know that x is a part of z, and no such information is here given to us. But the formula resolves the fraction into $\bar{x}z+\frac{0}{0}xz$. How is this? The reader will very likely guess at the solution for himself, especially if he appeals to the help of a diagram. But we must defer the explicit discussion of it until we have entered more thoroughly into the meaning and interpretation of logical equations in the next two chapters. For the present therefore much of what we say must be considered to be limited to those expressions in which both numerator and denominator could be interpreted if they stood alone. They probably form the great majority of the logical fractions we shall encounter in practice.

The reader will remember, in accordance with what was said in Chap. III., that when X and Y are each of them strict class expressions there is no need to take the trouble of analyzing them into all their component elements. We may write down at once $\frac{X}{Y}=XY+\frac{0}{0}\bar{X}\bar{Y}$, or $=X+\frac{0}{0}\bar{Y}$, whichever, under the circumstances, happens to be the simplest or easiest to obtain. (We must, of course, remember the implied condition of interpretability, $X\bar{Y}=0$). Thus $\frac{x+\bar{x}y}{z+\bar{z}w}$

involving, as it does, four terms, would expand by strict employment of the formula into 16 elements. This would be troublesome enough, and, what is worse, the process of 1 and 0 substitution is so intricate that we should be very likely to make a slip in performing it. So it is well to remember that we may write it down at once in the latter form, as

$$x + \bar{x}y + \tfrac{0}{0}\,\bar{z}\bar{w}.$$

Of course we must remember to add the implied condition (corresponding to $X\bar{Y} = 0$) which here becomes

$$(x + \bar{x}y)(1 - z - \bar{z}w) = 0$$

or $$(x + \bar{x}y)\,\bar{z}\bar{w} = 0, \text{ viz. } x\bar{z}\bar{w} = 0,\ \bar{x}y\bar{z}\bar{w} = 0.$$

In the development of $\dfrac{x}{y}$ we found that the only numerical multipliers of the various class elements which showed themselves were 1, 0, $\tfrac{0}{0}$ and $\tfrac{1}{0}$;—each of these being equivalent to a representative of a class, or a direction to take or leave a class. It may be enquired here whether there are any other possible factors besides these, or whether we have thus got specimens of all the possible factors? For an answer we must look to the process by which these results were obtained.

(1) First take the case in which we start with a directly interpretable class group, say for instance $x + y - xy$. What sort of numerical factors for our class terms can we get in this case? Our only means of getting such factors, remember, is by putting x and y, each in turn, equal to 1 and 0 (these factors being given by substitution in such expressions as $f(1, 1)\,x$, as described above). It is clear at once that such a process can yield us 1 and 0, and cannot yield $\tfrac{0}{0}$ and $\tfrac{1}{0}$. And if the class-expressions to which it is applied are in their strict shape, that is, if their terms were all mutually exclusive,

it can *only* yield 1 and 0; for any substitutional arrangement of 1 and 0 for x and y, which caused one of these terms not to vanish, would cause every other one to vanish. This follows from the very meaning, symbolically, of mutual exclusiveness.

But, if their terms are not mutually exclusive, then, two or more of them overlapping, we may get such factors as 2, 3, 4, or in fact any factor higher than 1. Thus, e.g., if we develop $x + y + xy$ (which may be taken as a redundant expression for 'x or y or both'), in which the element xy is counted three times over, we should find this definitely pointed out in the results of the development, by xy showing as $3xy$. Whereas, in the development of $x + y - xy$, which can be so arranged that the terms *are* mutually exclusive $(x + \bar{x}y)$, the factor of xy is unity; viz. it occurs as simply xy.

We are led then to this conclusion. Expand any expression which consists of a group of class terms; and if this expression is such that it can be thrown into the form of an aggregate of mutually exclusive and positive class terms, then the ultimate elements yielded by the development will either appear singly, or vanish entirely. That is, they will show only the factors 1 and 0. But if they were not mutually exclusive to begin with, and could not be arranged so as to be so, then the elements must appear affected with other signs than 1 and 0, such as -1, ± 2, ± 3, &c.: the negative signs showing that the original expressions were not merely badly phrased, but were unmeaning, by asking us to deduct when nothing was given from which to deduct. A moment's reflection upon the logical significance of the process, viz. that development or expansion is nothing but subdivision into ultimate elements, will show that these results are the only ones which could be expected or justified.

(2) Now consider the remaining, and only remaining case which it has been agreed to admit; that is, the inverse case when we have a fraction such as $\frac{x}{y}$ where the numerator and denominator are both of them class groups of the kind just described. The only difference here is that both this numerator and denominator can now yield those numerical factors, and no others, which either of them by itself could have yielded when it stood as a simple non-fractional class group. Naturally this lets in a larger range of possible numerical factors; for though the expressions x, y, and so forth, can only yield 1 and 0, it is clear that such an expression as $\frac{x}{y}$ can yield these, and also $\frac{0}{0}$ and $\frac{1}{0}$.

First consider the former of the two cases noticed above, viz. that in which each class group composing numerator and denominator is composed of mutually exclusive terms only. Here since each of these separately can (as above shown) yield us only 1 and 0, the combination of the two into a fractional form can yield us only the four forms 1, 0, $\frac{0}{0}$, $\frac{1}{0}$. These four were, as a matter of fact, yielded by the development of $\frac{x}{y}$.

Secondly, when either or both of these numerators and denominators had non-exclusive terms entering into it; we saw that, in addition to furnishing such numerical factors to the class elements as 1 and 0, they might also furnish any other numerical factor as well, according to the degree and nature of their defective statement. The only complication that is thus produced is that the final elements or subdivisions of such a fractional form may possess other numerical factors, positive or negative, besides the typical

four 1, 0, $\frac{0}{0}$, $\frac{1}{0}$. We shall have to take some notice of such results as these hereafter.

If it be asked here, as it very fairly may as a purely logical question, *why* the two peculiar factors $\frac{0}{0}$ and $\frac{1}{0}$ thus come into existence when we develop fractional forms, but not when we develop what may be called, by contrast, integral forms, the answer will readily be found by examining into the process by which they are obtained.

The answer is this: When we are developing an integral form, that is, a mere group of class terms, there can be no *indefiniteness* about this proceeding. Now $\frac{0}{0}$ is the sign of entire indefiniteness, and cannot therefore be required here. Of course if there was an indefinite term in the original class group it will reappear in the subdivision:—thus, for instance, $x + \frac{0}{0}y$ would clearly yield terms with this indefinite symbol prefixed to them:—but if the original group was definite it can only be divisible into one set of ultimate class elements, every one of which must necessarily appear. There can clearly be no opening for uncertainty about a true process of dichotomy. Again, as regards $\frac{1}{0}$. This we saw implied a proposition or statement. It gave an independent relation between the class elements, declaring that such a class was non-existent. For anything of this sort also there can be no opening in a mere class or group of classes.

With fractional forms the case is of course different. In whatever way $\frac{x}{y}$ be reached;—whether as a direct proposal indicated in that shape, or as obtained indirectly from such a statement as $zy = x$;—it makes a postulate. It accepts x and y under a condition, and this condition can of course be stated in a proposition. But every proposition is a material limitation of the formal possibilities of expression;

it extinguishes one or more of the classes yielded by mere dichotomy or subdivision. Thus though xy is a mere class, $\frac{x}{y}$ is a class with a condition attached; and it is this condition which, leading to the abolition of one or more classes, gives an opening for the directive symbol $\frac{1}{0}$ which demands the suppression of such classes.

So with the other symbol $\frac{0}{0}$. Our fractional form $\frac{x}{y}$ indicates an inverse operation, and one which, as customary in such cases, does not lead only to a single definite answer. The reply may very likely be that, within assigned limits, *any* class will answer our purpose. Such an indefinite symbol is therefore distinctly necessary.

It should be noticed that in exceptional cases no demand will be felt for either of these peculiar symbols, the conditions of the problem not giving occasion to them. Thus take the expression $\frac{xy + x\bar{y}z}{x}$. On expansion this yields $xy + x\bar{y}z + \frac{0}{0}\bar{x}$, but the usual condition (corresponding to $X\bar{Y} = 0$, in the development of the simpler fractional expression $\frac{X}{Y}$) gives here $(xy + x\bar{y}z)\,\bar{x} = 0$, which is identically true. That is, no further condition of relation between the symbols is demanded for the performance of the operation in question. The logical explanation of this is simply that x *includes* $xy + x\bar{y}z$, formally and without further postulate. Hence it must always be possible to find a class such that its limitation by x shall reduce it to $xy + x\bar{y}z$, which is the inverse operation called for. If it be known that Y includes X, we can always find a multiplier (that is, in logical language, some principle of selection,) which shall reduce Y to X.

Again, as regards the possible absence of terms affected by the indefinite sign $\frac{0}{0}$, take the following example:—

$$\frac{xy + x\bar{y} + \bar{x}\bar{y}}{xy + \bar{x}y + \bar{x}\bar{y}}.$$

Expand it; and it becomes simply $xy + \bar{x}\bar{y}$, the term which would have been affected by $\frac{0}{0}$ vanishing formally. The full logical explanation of this had better be deferred until we have adequately discussed the meaning and interpretation of logical equations, but the symbolic conditions for its occurrence are easily seen. The indefinite term in $\frac{X}{Y}$ is of course $\frac{0}{0}\bar{X}\bar{Y}$; it will therefore vanish whenever $\bar{X}\bar{Y} = 0$. That is, if X and Y together make up, or more than make up, the total universe (for this is the meaning of $\bar{X}\bar{Y} = 0$) then the inverse operation ceases to be indefinite[1].

This peculiar symbol $\frac{0}{0}$, perhaps the most distinctive feature in the Boolian Logic, has been strongly objected to. We shall have to discuss it again on several future occasions, but enough has now been said to show that it is very far from being borrowed in needless affectation from the mathematicians. If we chose to reject it we should still have to invent some symbol to take its place, and where else is such a symbol to be found which shall really express the full indefiniteness we need? There is no such word in the ordinary logical vocabulary, for '*some*' in both its senses ('some, it may be all', and 'some, but not all') excludes *none*, and therefore will not answer our purpose. We saw indeed, in Chap. VII., that $x = xy$ is really, though not very obviously,

[1] In which case, indeed, this inverse operation loses all peculiar significance. For, if $\bar{X}\bar{Y} = 0$, as well as $X\bar{Y} = 0$, then $Y = 1$; so that $\frac{X}{Y} = XY$ or $\frac{X}{1} = X$, which is identically true whatever may be the value of X.

the precise equivalent of $x = \frac{0}{0}y$; but we could not conveniently work with this equivalent in our developments, for it demands the repetition of the whole of one side of the equation on the other side. If we tried to employ it as a substitute for $\frac{0}{0}$ in the development of $\frac{x}{y}$, viz. $xy + \frac{0}{0}\bar{x}\bar{y}$, we should have to write it $\frac{x}{y} = xy + \frac{x}{y} \times \bar{x}\bar{y}$, when the impropriety of substituting an implicit equation instead of an explicit becomes glaring.

The reason why we require this symbol, whereas the ordinary Logic manages to do without it, must be assigned to the completeness of the Symbolic System. The ordinary Logic answers what it can, and simply lets alone what it cannot answer, whilst what the more general system aims at is to specify the direction and amount of our ignorance, in relation to the given data, as explicitly as that of our knowledge. We take into consideration every class which any combination of the terms in question can yield, and enquire what information the data will furnish in reference to it. As Prof. G. B. Halsted has well said, "The problem may be very compactly stated:...It is: Given any assertions, to determine precisely what they affirm, precisely what they deny, and precisely what they leave in doubt, separately and jointly"[1].

We are now in a position finally to explain the various elements which make their appearance in the development of one of these fractional forms which demand the performance of an inverse operation. If our data have been strictly

[1] *Journal of Speculative Philosophy*, Jan. 1878. See also the same Journal for Oct. 1878, and Jan. 1879, in each of which there are some valuable critical remarks by the same author on Boole's system.

expressed in mutually exclusive terms at the outset, our result must necessarily be comprised in the following form :—

$$X = A + 0B + \tfrac{0}{0}C + \tfrac{1}{0}D,$$

where X, A, B, C, D are aggregates of class elements composed of the various combinations of the class terms x, y, z, &c., and their contradictories.

Here then X, on the left side, is a class term. The succession of expressions on the right side may be taken as yielding a description or definition of X in terms of x, y, z, &c. Now what we are told in this equation is that, in order fully to determine X as desired, we must ;—

1. Take the whole of A ; A consisting of one or more class elements. Hence conversely, the whole of A is included in X. Ordinary Logic would express this by saying, 'Some X is A', and 'All A is X'.

2. Exclude from X the whole of the class B :—in ordinary language, 'No X is B'.

3. As to the entry of the class C into X ; we can, from the given data, determine nothing whatever. All C, or some only, or none of it, may go to the making up of X. To decide this, we should require fresh information.

4. As to D, this class is not merely excluded from X, as B was ; but it may be inferred from the given data that D is an impossible or non-existent class.

There are two questions which may fairly be raised here, and which demand a few moments' consideration.

It may possibly be objected that though we have thus determined the relation of X to the four classes A, B, C, D and to all the sub-classes which *they* contain under them, there may yet be other classes, of a similar kind to these, of which we have taken no account, for the possible combinations of x, y, z, &c. and their contradictories are

numerous. The reply is that there *are* no other classes possible, for however numerous they may be they are all comprised in A, B, C and D. The very nature of Development insures that every possible combination (logically) of our class terms shall be taken due notice of.

But, again, it may be asked, What about the *existence* of these classes referred to in A, B, and C? Must there be things corresponding to these various class terms? The answer to this question is contained in the results of a former chapter (Chap. VI.) and is briefly this. Negation is positive and final; therefore every one of the classes which go to make up D is certainly abolished: there can be no such things as these. Moreover we know for certain (on formal grounds) that some one class at least of those included in A, B, and C, must exist. But beyond this we can only speak hypothetically, as in all categorical universal assertion. If there are any A's then the same things must also be X, thus establishing that there must then be X. The existence of a B, however, or a C, proves nothing about there being X. Conversely, if there be any X, the things which are X must be either A or C, so that one of these is proved to exist, but we cannot tell which.

This process of Development suggests certain formulæ for the contradictories of class expressions, of which use will constantly have to be made.

1. The full contradictory of any given class expression may be defined as comprising 'all the rest' required to make up the universe with which we are concerned. Thus, in the simplest case, x and $\bar{x}$ are contradictories, because $x + \bar{x} = 1$. The complete theoretic process, therefore, for assigning the contradictory of any class expression involving two, three, four, &c. terms, would be to develop unity into its four, eight, sixteen, &c., elements, and then subtract

from this the given expression. The remainder is the contradiction required.

2. A modification of this complete process will often simplify matters. Suppose we have an expression of the form $Ax + B\bar{x}$, A and B being functions of other class terms, y, z, &c. Since $A + \bar{A} = 1$, and $B + \bar{B} = 1$, it is obvious that the contradictory of $Ax + B\bar{x}$ is $\bar{A}x + \bar{B}\bar{x}$. This requires only A and B to be contradicted separately, which will often be a much simpler matter.

This may be extended. Thus the contradictory of $Axy + Bx\bar{y} + C\bar{x}y + D\bar{x}\bar{y}$ will be $\bar{A}xy + \bar{B}x\bar{y} + \bar{C}\bar{x}y + \bar{D}\bar{x}\bar{y}$. As A, B, C, D, will often consist of single letters, the detailed contradiction will generally be obtained more simply than by the full operation described above. The only caution to be kept in mind in resorting to this process concerns the limiting values of A, B, C, D. If any one of these is *unity*, then, the contradiction of 1 being 0, the corresponding term has to be simply omitted. Thus if $A = 1$, the term in the contradiction involving xy is left out. Similarly if any one of these factors is 0,—i.e. if it did not appear in the original expression,—the corresponding element in the contradiction will appear simply. Thus if $B = 0$, the component involving $x\bar{y}$ will appear simply as $x\bar{y}$. For instance the contradiction of

$$zxy + \bar{x}y + wz\bar{x}\bar{y}$$

may be written down at once

$$\bar{z}xy + x\bar{y} + (\bar{w} + \bar{z})\bar{x}\bar{y}.$$

3. The safest, and generally the most convenient process in actual practice, consists of the extension of a rule given in its simple form by De Morgan:—For every elementary term substitute the contradictory, interchanging at the same time

the symbols of addition and multiplication. The elementary case here is that $\overline{xy} = \bar{x} + \bar{y}$, and $\overline{x+y} = \bar{x}\bar{y}$. This follows at once from (1); for by subtracting xy from $xy + x\bar{y} + \bar{x}y + \bar{x}\bar{y}$ we have $\bar{x} + x\bar{y}$, which is briefly (unexclusively) written $\bar{x} + \bar{y}$. Similarly, the expansion of $x + y$ being $xy + x\bar{y} + \bar{x}y$, if we deduct this from the full expansion of 1, we have $\bar{x}\bar{y}$.

This is, in most cases, the readiest process of contradiction.

We may illustrate all three processes by the expression

$$xy\bar{z} + x\bar{y}z + \bar{x}yz.$$

1. Expand 1 into its eight terms, and deduct the above three from the result. We have

$$xyz + x\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}.$$

2. Treating this as a case of $Axy + Bx\bar{y} + C\bar{x}y + D\bar{x}\bar{y}$, we have, in slightly different form,

$$xyz + x\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}.$$

3. The rule gives

$$(\bar{x} + \bar{y} + z)(\bar{x} + y + \bar{z})(x + \bar{y} + \bar{z}),$$

which, on multiplying out, yields

$$xyz + x\bar{y}\bar{z} + \bar{x}\bar{y} + \bar{y}\bar{z} + \bar{x}\bar{z} + \bar{y}\bar{z},$$

or, omitting included terms,

$$xyz + \bar{x}\bar{y} + \bar{x}\bar{z} + \bar{y}\bar{z}.$$

All these results will readily be found to be exactly equivalent.

We may take as a simple concrete example the following.

Express the class of persons who are *not* male guardians, nor female ratepayers, nor lodgers who are neither guardians nor ratepayers. How is the result affected if we know that all guardians are ratepayers, that every person who is not a

lodger is either a guardian or a ratepayer, and that all male ratepayers are guardians?

Put x for guardians, y and $\bar{y}$ for male and female, z for ratepayers, and w for lodgers, the class to be contradicted is

$$xy + \bar{y}z + w\bar{x}\bar{z}.$$

The elaborate process would be to develope this into its ten ultimate elements, and then subtract these from the 16 elements obtained by developing 1 in respect of x, y, z, w. By rule (3) we write down the contradiction at once as

$$(\bar{x} + \bar{y})(y + \bar{z})(\bar{w} + x + z),$$

and multiply out, obtaining

$$\bar{x}y\bar{w} + \bar{x}yz + \bar{y}\bar{z}\bar{w} + x\bar{y}\bar{z},$$

or $$\bar{x}y(\bar{w} + z) + \bar{y}\bar{z}(\bar{w} + x),$$

viz. males who are not guardians, whether ratepayers or not lodgers; and females who are not ratepayers, whether guardians or not lodgers.

If we revise this expression in accordance with the data $x\bar{z} = 0$, $\bar{w}\bar{x}\bar{z} = 0$, $yz\bar{x} = 0$, we shall find that it vanishes. That is the original expression contains the *whole* population in the Universe in question.

We may conclude this chapter by the consideration of some of the extreme and limiting cases which occur in the use of the full formula of expansion, as on p. 279.

Begin with one class term only, x, and see how many forms we can obtain with our $f(x) = f(1)x + f(0)\bar{x}$. Of what we have called integral forms there are the following four:— 1, 0, x, $\bar{x}$; which yield the developments,

$$1 = x + \bar{x}$$
$$0 = 0x + 0\bar{x}$$
$$x = x + 0\bar{x}$$
$$\bar{x} = 0x + \bar{x}.$$

These stand in need of no explanation. Logically, they represent the results of subdividing (or, we should rather say, attempting to subdivide) the four classes, 'everything', 'nothing', 'x', and 'not-x', into their x and not-x parts respectively. Of course only 'everything' can really be so divided.

Of fractional forms with one class term, we have sixteen, since any one of the above four may stand as either numerator or denominator. Of these however the four with unity as denominator are identical with the above integral forms, thus leaving twelve. They should all be carefully worked through for the purpose of obtaining full command of the logical meaning and symbolic usage of our various forms. They are as follows:—

1. $\frac{1}{x} = x + \frac{1}{0}\bar{x}$.	7. $\frac{1}{0} = \frac{1}{0}x + \frac{1}{0}\bar{x}$.
2. $\frac{x}{x} = x + \frac{0}{0}\bar{x}$.	8. $\frac{\bar{x}}{0} = \frac{0}{0}x + \frac{1}{0}\bar{x}$.
3. $\frac{0}{x} = 0x + \frac{0}{0}\bar{x}$.	9. $\frac{1}{\bar{x}} = \frac{1}{0}x + \bar{x}$.
4. $\frac{\bar{x}}{x} = 0x + \frac{1}{0}\bar{x}$.	10. $\frac{x}{\bar{x}} = \frac{1}{0}x + 0\bar{x}$.
5. $\frac{x}{0} = \frac{1}{0}x + \frac{0}{0}\bar{x}$.	11. $\frac{0}{\bar{x}} = \frac{0}{0}x + 0\bar{x}$.
6. $\frac{0}{0} = \frac{0}{0}x + \frac{0}{0}\bar{x}$.	12. $\frac{\bar{x}}{\bar{x}} = \frac{0}{0}x + \bar{x}$.

We will briefly examine a few of these results in turn; sometimes in accordance with one, sometimes in accordance with the other, of the two ways already pointed out for approaching these fractional forms; that is, either by taking them as immediate demands for the performance of an inverse process, or as stating a conditional proposition which leads to such a demand.

1. This asks us the question, What is that class which, when restricted by taking only the x-part of it, will yield the class '*all*'? A little reflection will show that the only class of which this can be said *is* 'all'; viz. this is the limiting case in which the imposition of a restriction is merely, as it happens, leaving the class unaltered. (The formula indicates this by the term $\frac{1}{0}\bar{x}$; reminding us that $\bar{x}=0$, viz. $1-x=0$ or $x=1$.)

2. This is in reality a very old friend of every logician, for it is nothing else than 'accidental conversion' done up in a new dress, and, I would add, more accurately expressed. No doubt it sounds very unfamiliar, even when turned from symbols into words, if we phrase it as 'What is the most general expression of that class which, restricted by taking only the x-part of it, will coincide with x?' But we shall soon recognize its features if we look at it in this way:— Suppose we had given us that $yx=x$, and were asked, What then is y in general? This would have led to $y=\frac{x}{x}$, as above, which is asking for y in terms of x. Now an expression for the universal proposition 'All x is y' is $x=xy$. Hence to ask for y in terms of x, under this condition (as is done symbolically above), and to ask 'If all x is y, what is y?' are really one and the same question.

Now compare our solution here with the common solution. We say that $y=x+\frac{0}{0}\bar{x}$, whilst the common answer says simply 'Some y is x'. The advantage of the former seems to be that it forces on our attention (by the employment of the peculiar symbol $\frac{0}{0}$) several possible cases which the common answer rather tends to obscure from sight. We are reminded for instance that x may be the whole of y (if $\frac{0}{0}$ happens to $=0$); that it may be a part only (if $\frac{0}{0}$ is intermediate between 0 and 1); or that y itself may be 'all'

(if $\frac{0}{0}=1$). These three alternatives may of course be deduced from the common form of conclusion, but they certainly do not appear very prominently in it, especially considering the ambiguity of the word *some.*

I would also call the reader's attention to the very decisive way in which all those troublesome and perplexing questions, as to what is implicated in the way of the existence of our x and y, are avoided by this way of regarding the subject. Once understand that 'all x is y' is only unconditional in what it *denies* (i.e. in denying that there is any $x\bar{y}$); and employ the truly indefinite symbol $\frac{0}{0}$ instead of *some,* and a proposition and its converse will fit in harmoniously with any number of other propositions without inconsistency or demand for fresh assumption. For instance: start with 'all x is y' in the form $x\bar{y}=0$, thus blotting out one class. Now elicit x from this, and we get $x=\frac{0}{0}y$ (the reader will easily verify this conclusion as follows: $x=\dfrac{0}{\bar{y}}$, therefore by development $x=\frac{0}{0}y$). The implication is clear and decisive, and in perfect harmony with the unconditional negation above. We see that if there be x there must be y, and that if there be y there may be x; but that there may be neither one nor the other. Then proceed to convert $x\bar{y}=0$, or $x=\frac{0}{0}y$, and we get, as above, $y=x+\frac{0}{0}\bar{x}$. Here the same implications meet us as clearly as before:—if there be any x there must be y, but if there be any y there need not be x (since the y may be contributed by the term $\frac{0}{0}\bar{x}$). And there need not be either x or y.

3. This result is best interpreted as follows. Suppose $xy=0$, viz. 'No x is y', What do we know about y? i.e. what is our account of Conversion or Contraposition? The equation $y=\frac{0}{0}\bar{x}$ tells us that y, if it exist, must lie outside x, but that there may be no y at all (if $\frac{0}{0}=0$); that there may

be some y; or that the y may be the whole of $\bar{x}$ (if $\frac{0}{0}=1$), in which case x and y are contradictory opposites.

4. This development must have come about from such a statement as that 'xy is the same as not-x'. We can save this from being nonsense only by supposing that there is no such class as y, and that x is 'all'; for 'all nothing' is the same as 'no all'. The development expresses this condition.

6. This is the development of an entirely indefinite class, and is therefore itself entirely indefinite. The nearest verbal equivalent would be the direction to subdivide 'something' into its x and not-x portions. We could but say that this would yield some x and some not-x. After what has been already said in explanation of this symbol $\frac{0}{0}$ the reader will hardly need to be reminded that we regard it as a strictly interpretable expression. It is the determination (if we may use the word) of a class which proves to be strictly indeterminable.

7. This is of course an instance of gross misapplication of our formula. The Universe being 1, the expression $\frac{1}{0}$ is logically unmeaning as a class term. The demand which it makes is that we shall find some class such that when *no* part of it is taken we shall still have, as a result, *all*; $(0x=1)$.

What the formula replies to this apparently absurd proposal is that, under these conditions there can be neither x nor not-x. Has this reply any significance? It appears to me to point to the same result as we have already discussed (on page 166), viz. the result in which the 'universe' itself shrinks up to *nothing*. The expression $x+\bar{x}$ must always be equated to 1; but when we go behind the symbols, and ask what this figure 1 denotes, we must recognize that the significance remains unchanged whatever the range of this denotation may be. Suppose that our Universe continually diminishes, and at last vanishes, then x and $\bar{x}$ will both tend

to disappear. And if under these circumstances we say that they do both disappear 'in the limit', we are only following recognized mathematical usage. Of course their whole significance, as representations of classes, has by then been lost. The rational interpretation therefore of the formula, in this extreme case, seems to be that by diminishing our Universe we can make both x and $\bar{x}$ as small as we please; and this without any limit whatever short of actual zero.

CHAPTER XI.

LOGICAL STATEMENTS OR EQUATIONS.

HAVING thus considered the nature of Development or Subdivision, which may be considered an introduction to the central subject of Logic, we must now go on to consider this main subject itself under the heads of *Logical Equations*, and *Interpretation and Solution of Logical Equations*. This latter division may be said, roughly speaking, to correspond to that between Propositions and Reasonings, in ordinary Logic, though here as elsewhere our arrangement of the subject is very far from coinciding with the traditional one. We shall devote this and the following chapters to the consideration of these topics.

There are two main principles of interpretation to which we shall have to appeal in the course of this discussion. As neither of them is distinctly recognized in the ordinary Logic, and both are in some respects decidedly alien to the ways of common thought and speech, it will be well to begin by calling prominent attention to them.

(1) The first of these is involved in the view of the Import of Propositions explained and insisted on in the sixth chapter. The comparative novelty of that view as a

systematic doctrine, and its extreme importance for our present purpose, must be the excuse for once more recalling the reader's attention to it.

It was laid down then that propositions must be regarded as having, generally speaking, an affirmative interpretation of a conditional and somewhat complex character, and a negative interpretation which is unconditional and comparatively simple. That is, what they assert can only be accepted under hypotheses and provisionally, in so far as the existence of the objects is concerned, whereas what they can be made to deny is denied absolutely. This contrast presented itself even in the case of the simple propositions of the common Logic, but when we come to the complicated systems of propositions which we must be prepared to grapple with in Symbolic Logic, it appears to me that without this explanation we can make no way at all.

It must be frankly admitted that this is not the sense in which the popular mind accepts and interprets propositions. Nor is it entirely in accordance with the canons of the common Logic;—and very reasonably not so, on the part of the latter; for using, as this does, forms which are but little removed from those of common speech, it cannot risk so complete a breach with convention as they may freely do who deal mostly with symbols.

(2) The other principle to which we shall have to resort may be conveniently introduced by the following question. If any one were to declare to us that his annual money income and the acreage of his landed estate, taken together, amounted to precisely £500 and his daughters, could we charge him on the face of the matter with either falsity or nonsense? He has adopted an extremely unusual way of speaking, but a solecism need not be without meaning. If we insisted on translating his words strictly, what sort of

construction should we have to put upon them, neglecting all merely conventional implications? We should have to say that, since it is impossible to equate heterogeneous things, the only solution which will avoid actual contradiction must be found in the conclusion that his income is £500, and that he has no acres and no daughters[1].

The second of these principles involves of course an appeal to the first. Admitting that the employment of a logical term does not necessarily carry with it the existence of any corresponding class, we say that there are circumstances in the case in point under which this admission has to be put in force. Disparate things can only be equated by the assumption that both are then and there non-existent.

There is a slightly different way of looking at the same conclusion which may make it somewhat more acceptable. Instead of starting, as above, with an equation $x = y$, where x and y are heterogeneous, put the statement in the form that if y be taken from x nothing is left. Suppose I say, in reference to some assemblage of people, Take all the rogues from amongst the bookmakers and nobody is left, it is quite certain that this identifies the two classes. This is a necessity of thought or of things; but to this necessity common usage couples an assumption, which ordinary Logic doubtfully accepts, viz. that we must not so speak unless we mean to imply that there are certainly some people present who belong to both categories. Symbolic Logic distinctly rejecting this assumption need not hesitate to accept at the same time the proposition that no bookmakers are rogues. In this case, since the subduction of a rogue can no longer remove a bookmaker the statement can only hold good on

[1] There was an old saying at Croyland in the fens,—then inaccessible to wheeled traffic—, that all the carts that came there had the tires of their wheels of silver.

the supposition that no member of either class is present. In other words, whereas the logical equation, $x - y = 0$, necessarily and always implies the identity of x and y, all that we are now doing is to claim the right of extending this implication to the limiting case in which x and y are both $= 0$; and consequently of inferring that if x and y are known to be mutually exclusive, then this limiting case is the only possible one.

It can hardly be maintained that this way of looking at the matter is much more in accordance with popular convention than the former. The complaint would presumably be, not that we were misinterpreting statements, but that we were displaying a pedantic and lawyer-like[1] determination to insist upon an interpretation of a statement which was absurdly framed. Both the principles above enunciated are certainly legitimate. No one can say that they actually contradict or defy any known law of thought, or any express enactment of Logic: the latter of them indeed errs, if at all, from excessive adherence to the law.

But it will be readily seen that the principle we have thus invoked is of wider application. It is not merely true that the statement $x - y = 0$, when x and y are exclusives, leads to $x = 0$, $y = 0$. The same holds good when such mutual exclusives are combined by way either of addition or subtraction, and when each of them is multiplied by any numerical factors whatever. That is to say, such an expression as $ax \pm by \pm cz \pm$ &c. $= 0$, (so long as x, y, z, &c. are

[1] Not I presume that the lawyers would so determine such a question. De Morgan (*Syllabus*, p. 12) gives the following example on a somewhat analogous point: "An Act was once passed exempting indentures [of apprenticeship] from duty when the premium was under five pounds sterling: the Court of King's Bench held that the exemption did not apply when there was *no premium at all*, because 'no premium at all' is not 'a premium under five pounds.'"

exclusive class terms) necessarily leads to $x=0$, $y=0$, $z=0$. This follows from the very meaning and employment of our class terms. Really independent, *i.e.* exclusive, classes are absolutely powerless upon each other; it is not possible for one to neutralize another and thus to offer a substitute for the common extinction of all. We must not be misled by the analogy of ordinary arithmetic, where we add and subtract magnitudes, and where in consequence there are many different ways of adjusting the contributions of the separate items in a total. If any analogy is sought in that direction it must be in the equation to zero of the sum of a number of squares, which involves the separate equation to zero of each element of the total[1].

It will now be seen therefore, that in order to analyze a logical statement, and to extract from it the sum-total of what it has to tell us, all that is necessary is to break it up into a series of terms the sum of which is declared equal to zero[2]. The information yielded by the statement can then be read off at once in all its details, in the form of a number of separate denials. This, it may be pointed out, is the full analytical process; the subsequent synthetic process, which seeks to build up these details into new forms and thus fully to interpret them, will have to be discussed in a future chapter. The full importance, from a theoretic point of view, of the Rule of Development explained in the last chapter will therefore be apparent. The desired result,

[1] A still more apposite analogy is offered here in the Science of Quaternions. The single equation $xi+yj+zk=\xi i+\eta j+\zeta k$ does not there lead to the indeterminateness of an ordinary algebraic equation, but necessitates separately;

$$x=\xi,\quad y=\eta,\quad z=\zeta.$$

And the reason is the same, viz. that we are comparing heterogeneous or disparate things in i, j, k.

[2] If the terms are mutually exclusive, then, as indicated above, we might say the same of either a sum or difference.

viz. of securing that some complicated expression shall be broken up into all its ultimate and consequently mutually exclusive elements, is precisely that for which this Rule is devised.

I. Take the simple case of an explicit statement, by which is here meant one in which we have only a single term standing by itself on one side, this being equated, on the other side, to a group of terms. For instance,

$$w = xyz + \bar{x}y + x\bar{y}\bar{z} \quad\text{.................(1).}$$

We here have a description, definition, or synonyme of any kind (neglecting, as we do, all but the denotative import of our terms, these various expressions are regarded as equivalent) of w in terms of x, y, and z. The individuals referred to by w are identical with the aggregate of the individuals referred to by the three terms equated to w. Of course as regards the expression or extension of our knowledge by such a statement, various views may be taken. If all the three terms on the right hand are known, then w may be known thereby. If w were known, then we have one condition assigned by which to determine the other elements. If we happen to be equally familiar with both, then the equation may be regarded as a statement of our knowledge.

As regards the verbal statement of such a proposition, it is a matter of choice whether we throw it into the categorical, hypothetical or disjunctive form. There may be no such things really as w, xyz, $\bar{x}y$, &c.; or some only of these classes may be missing. If we want to avoid any reference to such contingencies,—as common logic mostly does,—then we should put it either collectively by saying, 'the w's are identical with the sum of the things which are xyz, or y and not-x, or x and neither y nor z'; or distributively 'every w

is either xyz, y and not-x, or x and neither y nor z'; and conversely. If we wish to indicate that our terms only hold their life, so to say, subject to the conditions entailed by other propositions which may be impending, then we might say '*if* there be any w it must be either xyz, &c.' The popular expressions which thus cover the ground of a single symbolic statement are very various.

So much for the affirmative interpretations: now look at the negative interpretation. The statement was that the whole of w was confined to a certain number of compartments, and conversely that every class of things occupying any of these compartments must be a w. Now, on the system of making a perfectly exhaustive scheme of classification of our various class terms, it is clear that to say that w is *within* certain compartments is precisely equivalent to saying that it is *without* certain others. Hence it follows that an alternative or disjunctive affirmative can be broken up into a number of independent negatives. This of course is in no way peculiar to our system; for every one knows that the proposition 'All X is either Y or Z' may be phrased 'no X is (neither Y nor Z)'. What is characteristic of this symbolic Logic is the symmetry and generality with which this procedure is carried out.

Now look at this symbolically. What we have to do is to break up the given statement into a series of separate statements each expressing that such and such a combination $= 0$. There are a variety of ways of doing this. We might, of course, expand each part of the whole expression into its ultimate elements, in accordance with the suggestions at the commencement of this chapter. But in practice the most convenient plan generally is to multiply each side by the contradictory of the other, and add the results together. Suppose we had the very simple statement $x = y$ we should

throw this into the two negations $x\bar{y} = 0$, $\bar{x}y = 0$. Now the same holds true for aggregates of class terms as well as for simple ones, for such aggregation does not destroy or in any way affect their class characteristics. Hence the equation $w = xyz + \bar{x}y + x\bar{y}\bar{z}$, may be fully expressed negatively by the two

$$\left.\begin{array}{r}\bar{w}\,(xyz + \bar{x}y + x\bar{y}\bar{z}) = 0\\ w\,(1 - xyz - \bar{x}y - x\bar{y}\bar{z}) = 0\end{array}\right\}.$$

The upper of these is already in the form of a sum of negations, and therefore breaks up at once into three separate negations. The lower as it stands has one positive and three negative terms; but it may be easily put into a form composed of positive terms only, by either of the rules given at the end of the last chapter. The result is then

$$w\,(xy\bar{z} + x\bar{y}z + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}) = 0.$$

When this is added to the similar result in the former equation (those three, by the resolution of $\bar{x}y$ into $\bar{x}yz$ $\bar{x}y\bar{z}$, yield four ultimate terms) we have the eight following elementary negations:—

$$\begin{array}{ll}\bar{w}xyz = 0 & wxy\bar{z} = 0\\ \bar{w}\bar{x}yz = 0 & wx\bar{y}z = 0\\ \bar{w}\bar{x}y\bar{z} = 0 & w\bar{x}\bar{y}z = 0\\ \bar{w}x\bar{y}\bar{z} = 0 & w\bar{x}\bar{y}\bar{z} = 0.\end{array}$$

These eight denials contain every particle of information yielded by the original statement, or in any way deducible from it.

In the last case we had a single term on one side of the equation. As any combination of class terms is still representative of a logical class, there is no essential distinction introduced by having a group of terms on each side. Take, for instance,

$$x\bar{y} + \bar{x}z = \bar{x}y + \bar{y}z.$$

A very little observation will show that the two classes in question can only be made equal to each other upon the conditions expressed by $x\bar{y}\bar{z}=0$, $\bar{x}y\bar{z}=0$. On any other supposition we should be making an assertion equivalent to the declaration alluded to above, of the identity of the money income with the acreage or family. This is more obvious on subdividing the terms on each side, when they stand

$$x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}z = \bar{x}yz + \bar{x}y\bar{z} + x\bar{y}z + \bar{x}\bar{y}z$$

or $x\bar{y}\bar{z}=\bar{x}y\bar{z}$, which of course, as already described, demands the destruction of each of these two classes. Hence the full interpretation of the given statement, in so far as analysis is concerned, is given by these two elements,

$$\left.\begin{array}{r} x\bar{y}\bar{z} = 0 \\ \bar{x}y\bar{z} = 0 \end{array}\right\}.$$

Had we adopted the other plan, viz. that of multiplying each side by the contradictory of the other, we should have

$$(x\bar{y} + \bar{x}z)\,\overline{(\bar{x}y + \bar{y}z)} = 0$$
$$(\bar{x}y + \bar{y}z)\,\overline{(x\bar{y} + \bar{x}z)} = 0.$$

Developing, and multiplying out, by the rules of the last chapter, we have as before

$$x\bar{y}\bar{z} = 0, \quad \bar{x}y\bar{z} = 0.$$

II. In the cases hitherto discussed, all the terms entered *definitely* into the equations or propositions which involved them. We must now discuss the case in which one or more of the terms are affected by the *indefinite* sign $\frac{0}{0}$. Begin with the simplest of cases, in which a definite class is equated to an indefinite one, for instance

$$w = \tfrac{0}{0}xy.$$

This form was examined in Chap. VII. We showed that it is, with reserves and explanations, the most accurate symbolic equivalent for the somewhat ambiguous 'all w is xy' of ordinary Logic, viz. for a form of the Universal Affirmative. We also considered its negative equivalent in the same chapter, but we must now compare it somewhat more fully with the definite statements above considered. In those cases what we did was to exactly *identify* one class group with another, which gave rise to *two* negations: for we could deny of each group that it had any members outside the other. In the present case we merely say more vaguely that one group is contained somewhere within the other, which only gives rise to the single denial that the former or definite group has members outside the extreme limits of the latter or indefinite one. Hence all that can be elicited from such a form as the one now before us is,

$$w(1-xy)=0.$$

For xy is the extreme limit of the indefinite class $\frac{0}{0}xy$, when $\frac{0}{0}$ becomes $=1$. Hence we can assert unconditionally no more than that there can be no w which lies outside xy. Of course, if we like to do so, we can break up the expression, $1-xy$, into the constituent members of which it is composed, viz. $\bar{x}\bar{y}+\bar{x}y+x\bar{y}$. Then the equation resolves itself into three denials given by

$$\left.\begin{array}{r} wx\bar{y}=0 \\ w\bar{x}y=0 \\ w\bar{x}\bar{y}=0 \end{array}\right\}.$$

Now take a somewhat less simple example involving indefinite terms. Suppose we have

$$w=xyz+\bar{x}y+\tfrac{0}{0}x\bar{y}z.$$

First as regards its significance. This is not, like the

examples we began with, the exact identification of two class groups with each other, for the term $\frac{0}{0}x\bar{y}z$ comes in to prevent such identification. The equation cannot therefore be regarded as expressive of a definition or description of w. What exactly it tells us is this:—That w certainly comprises the whole of the two classes $xyz + \bar{x}y$ and that it may or may not take in the class $x\bar{y}z$. The indeterminate sign therefore is a sort of "look out" to us to be prepared for individuals from the class so affected. The whole of the class may be wanted; or, if it be subdivided, a part of it only, or possibly none at all may be wanted. This is left altogether indeterminate.

It is clear therefore that we cannot here make quite such a simple double negation as we did in the former cases. What we have to do instead is to take account of one limit of $\frac{0}{0}$ (viz. 1) in one denial, and of the other limit (viz. 0) in the other denial.

We may say with certainty that there is no w which lies outside $xyz + \bar{x}y + x\bar{y}z$, for this represents the extreme limit of the admissible indefiniteness; and we may say with similar certainty that there is no $xyz + \bar{x}y$ which lies outside w, for this represents the extreme limit of the indefiniteness in the opposite direction. These two statements yield us a pair of negations which do not quite so accurately balance each other as was the case when we were concerned with definite terms only, for one is less extensive than the other. Put into symbols they stand,

$$\left.\begin{array}{l} w(1 - xyz - \bar{x}y - x\bar{y}z) = 0 \\ \bar{w}(xyz + \bar{x}y) = 0. \end{array}\right\}.$$

We might then proceed, by expanding 1 in terms of x, y, and z, to convert the former into positive terms only, and should thus be finally left with a string of separate negations

as in the former cases; the only distinction being that owing to the occurrence of the $\frac{0}{0}$ term we get *fewer* of these unconditional negations than we should otherwise obtain, and therefore our materials of information are less abundant.

The difference thus marked in the symbols is equally noticeable in the verbal expression: that is, as regards our powers of conversion and contraposition where these indeterminate terms occur. What we *ought* to say is on the one hand, 'All w is made up of $xy\bar{z}$, $\bar{x}y$, and (possibly) $x\bar{y}z$', and on the other, 'All xyz and all $\bar{x}y$ are w'. What we may be tempted to say, however, is 'All w is made of xyz, $\bar{x}y$, and *some* $x\bar{y}z$', thus omitting the full indefiniteness of $\frac{0}{0}x\bar{y}z$.

As it is always well to examine limiting cases, we will turn to see what these superior and inferior limits of negation become when we have none but indefinite terms on one side. Recur to the example

$$w = \tfrac{0}{0}xy.$$

The limit of $\frac{0}{0}$ in one direction is 1; thus giving us the negation $w(1-xy)=0$, viz. that 'No w can lie outside xy'. But the limit in the other direction is 0. In this case the whole of the right side of the equation vanishes, and we can make no corresponding denial by means of this inferior limit; or rather, in formal strictness, such denial assumes the form 'No $0xy$ lies outside w', which tells us nothing whatever beyond the obvious fact that w cannot be less than 0. We are thus reminded again of the distinction between this really indefinite factor $\frac{0}{0}$, and both the *some*, and the undistributed predicate, of ordinary logic. These latter exclude the value 0, therefore we can always make something out of the statement 'All w is xy' in both directions. In one direction the result agrees with our symbolic expression $w(1-xy)=0$, viz. 'No w is not-xy'. In the other it is

generally stated positively, in the form 'Some (*i.e.* not none) xy is w'. The validity of such conversion has been already discussed in a former chapter.

Only one other case then remains, viz. that in which there is an indefinite term on each side, for instance[1]

$$x + \tfrac{0}{0}y = z + \tfrac{0}{0}w.$$

Adopting the same explanation as before, all we can now say is that there can be no x outside $z + w$, and no z outside $x + y$. Hence the full significance of the expression is given by the dénials $x\bar{z}\bar{w} = 0$, $z\bar{x}\bar{y} = 0$. Verbally the symbolic statement might be read off as the pair of propositions:—All x is z or w; all z is x or y.

If x and y, as also z and w, are respectively mutual exclusives, so that the conditions of fractional statement are satisfied, the expression becomes equivalent to $\dfrac{x}{\bar{\bar{y}}} = \dfrac{z}{\bar{\bar{w}}}$.

We may sum up, therefore, by saying that the introduction of indefinite terms into our equations successively limits the range of consequent denial, as illustrated by the three stages discussed above:—

$$\begin{array}{rcl} x = z & \text{leads to} & x\bar{z} = 0,\quad \bar{x}z = 0, \\ x = z + \tfrac{0}{0}w & ,, & x\bar{z}\bar{w} = 0,\quad \bar{x}z = 0, \\ x + \tfrac{0}{0}y = z + \tfrac{0}{0}w & ,, & x\bar{z}\bar{w} = 0,\quad \bar{x}\bar{y}z = 0. \end{array}$$

[1] An incorrect explanation of this case was given in the former edition. For a full discussion of such forms see Schröder, *Vorlesungen*, I. 533.

CHAPTER XII.

LOGICAL STATEMENTS OR EQUATIONS (*continued*).

HAVING thus explained the meaning and interpretation of a single logical statement or equation, we must now go on to discuss the results of a system of them. Propositions do not generally present themselves alone, but in groups of two or more, and we have already had repeated opportunities of examining the results of their combination in simple examples. But we must now give more definite consideration to the specific question whether a combination of equations differs in any essential way from a single equation; and, if it does, what are the nature, ground, and limits of such difference.

The principal misleading influences to which the reader will probably be exposed here are those introduced by the associations of mathematics. In mathematics a combination of equations differs from a single equation by what may almost be called a difference of kind rather than one of degree only. Generally speaking, a single equation which involves two variables, x and y for instance, admits of an infinite number of solutions; and the answer is so far left entirely indeterminate. When we introduce two equations,

so serious a restriction is introduced that we are at once tied down to a single and determinate value for x and y, if the equation were of the first degree. If we combine together more than two such equations, we should find them, generally speaking, in direct conflict with each other. The third, if not simply deducible from the other two, would be irreconcileable with them.

In the case of Logical equations however there is nothing resembling all this. There is no necessary difference between the nature and amount of information yielded by one, and that which we obtain from two. Given one equation, the addition of a second to it, involving the same terms, will no doubt, generally speaking, add to our information, but it will not do so to the marked and striking extent to which we are accustomed in mathematics. There is nothing here like the precise assignment of a single point by the intersection of two lines; the result is somewhat more analogous to the contraction of a circle into one of a smaller radius. We are still referred to a class of some kind, whether we are supplied with one equation or with many; but in the latter case the class is narrowed by the excision, probably, of a number of its various subdivisions.

If the reader recurs to that negative interpretation of a proposition which we have had so often to insist upon, he will see at once why this is so. Start with one equation: this will yield us a certain number of destructions of possible classes. How many it will thus destroy is of course dependent upon various considerations, such as the relative magnitude of the classes with which it deals. Some however it must certainly destroy, or it would be without any material significance. Now add on another equation. This will similarly proceed to make some further such clearances. Unless the second be a mere repetition of the first, or of

some part of the first, it must follow that the second will make some clearances amongst classes which had been left surviving by the first. It will in fact add to our actual knowledge. But this does not represent any intrinsic superiority in the information yielded by the two over that yielded by one; the sum-total of the information might equally well have been conveyed by one singly.

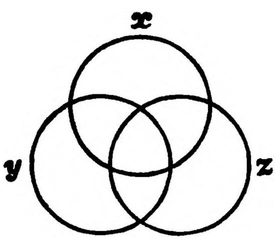
If any doubt is felt on this point, an appeal, symbolically and diagrammatically, to any simple example will serve to make it quite plain. Take the following examples;

$$\begin{cases} w = \frac{0}{0}(xy + \bar{x}z) \ldots\ldots\ldots\ldots (\alpha) \\ w = \frac{0}{0}(xyz + \bar{x}\bar{z}) \ldots\ldots\ldots\ldots (\beta) \end{cases}$$

They would be naturally expressed in words, by saying,

{All w is either x and y, or, if not x, then z.
{All w is either x, y, and z, or neither x nor z.

Begin by looking at the affirmative side of these two statements and their combination together.



Equation (α) asserts that the whole of the class w is confined to some portion of the two compartments xy and $\bar{x}z$. Similarly equation (β) confines the members of this same class w to some portion of the compartments xyz and $\bar{x}\bar{z}$. It is clear therefore that the two equations when taken in combination confine the class w to the one sub-division common to these two assignments, viz. to xyz. To put it into ordinary language, we are told that 'all w is xyz'.

In this case the class w was originally referred to two different groups of sub-classes which partially overlap each other, thus restricting w to their common part. It will readily be seen that this must represent the usual case. For failing this, what else could we have? Either the two groups to which w is referred must be entirely distinct from each other, which is tantamount to saying that there is no w at all; or the one group must be entirely contained within the other, in which case the narrower determination completely supersedes the broader.

Of course when, as here, we have a class w thus standing by itself, on one side, and on the other side a group of classes determining it with more or less definiteness, there is a much shorter method of procedure than that given above. All we have to do is to 'multiply' the two determinations together, when the common elements will stand out as the only survivors. Thus,

$$\begin{aligned} w &= \tfrac{0}{0}(xy + \bar{x}z)(xyz + \bar{x}\bar{z}) \\ &= \tfrac{0}{0}\,xyz. \end{aligned}$$

Had the two groups been identical, this product would have simply repeated either of them. Had they been entirely distinct, the result would have been exhibited in the shape $w = 0$.

The same results come out equally distinctly under the negative interpretation of our statements. In fact this is the simplest and most general way of regarding them; for explicit equations of the kind of which our present example is an instance are but one class of logical equations, whereas we have shown that all logical equations without exception are resolvable into a series of distinct negations. Under this interpretation we ask, What subdivisions does each equation blot out? instead of asking what classes it suffers to remain in existence.

By the methods of treatment described in the last chapter we should resolve each equation into all the ultimate denials which it involves, and which together constitute its effective meaning. They would then stand as follows;—

$$\begin{cases} w\,(x\bar{y}z + x\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z}) = 0 \\ w\,(\bar{x}\bar{y}z + x\bar{y}z + x\bar{y}\bar{z} + xy\bar{z} + \bar{x}yz) = 0. \end{cases}$$

Now denial, as we have seen, is unconditional and final. All therefore that we have to do is to add up these separate elements, lay aside those which merely repeat what has been already denied, and observe what number of classes they succeed in accounting for. In the case before us, two of the denials are twice repeated, thus indicating that, to that extent, the given equations were not altogether independent of each other. Hence the nine elementary negatives we have obtained are reduced to seven, leaving one and only one surviving, viz. $wxyz$. If this be put into words it may be phrased thus, 'The only w that can exist under the given conditions is the w that is xyz'. This is, as we know, a true interpretation of the familiar Universal Affirmative which was obtained by the other mode of treatment, 'All w is xyz'; so that the two views lead us to the same conclusion.

It appears therefore that the most which a second equation can do is to supplement the information afforded by the first; at least in instances of the kind which we have examined, and which may be taken as fair samples of those which we meet in Logic. As before remarked, there may not even be this supplementary information afforded. If the group of compartments to which the second equation refers us be wider than that assigned by the first, then we learn nothing new. Thus $w = \frac{0}{0}(1 - xyz)$ adds nothing whatever to the statement $w = \frac{0}{0}(1 - xy)$; since $(1 - xy)$ is contained in the class $(1 - xyz)$ and is therefore a narrower

determination than that which is offered to supplement and restrict it.

Conversely when the new assignment is a narrower group than the old we do not strictly speaking supplement our information; we rather supersede entirely the old assignment. Thus for instance if $w = \frac{0}{0}(1 - xyz)$ had been given to us first, this would be entirely superseded by the equation $w = \frac{0}{0}(1 - xy)$.

Hence the appropriate instances of combination of equations, of this explicit character, will be when the class determination of the one partially overlaps that of the other. Each then contributes something to the other, that is, each in part curtails the limits which the other had assigned. I recall attention to this point because it is so strongly characteristic of that class-view or denotation-view of the import of terms which necessarily pervades this account of Logic. In common Logic when we refer to xy, we as often as not think of x and y merely as attributes imposed upon an object. That is, we think of them in their *connotation*; and, so thinking of them, we make no reference to anything but the *presence* of such attributes:—we do not simultaneously impose the *absence* of these same attributes upon any other objects. We simply think nothing about such absence elsewhere. But when x and y are regarded as classes we cannot but observe that not-x and not-y are themselves just as much classes as those of which they are the contradictories. To say therefore that a thing is xy assigns it as a rule to two overlapping classes: the one assignment cuts off the class $x\bar{y}$, and the other cuts off $\bar{x}y$, leaving only xy out of the whole contents of x and y together.

We gather then that, except for purposes of mere arrangement, such as symmetry or brevity or clearness, it

is a matter of entire indifference in how many propositions or equations any given stock of logical information is conveyed. The full significance can always (unlike the case of mathematical equations) be conveyed in the form of a single equation which equates to zero the sum of a number of distinct alternatives.

For instance in the example towards the commencement of this chapter, we saw that all which the two equations had to say could be completely said by one. We might equally take a step in the opposite direction by increasing their number, that is, by breaking them up into smaller contingents. The two, as originally given to us[1], stood thus:—

$$\begin{cases} w = w\,(xy + \bar{x}z) \\ w = w\,(xyz + \bar{x}\bar{z} \end{cases}$$

Now develop each side of the upper one in respect of x, or, in more familiar words, break each side up into its x and its not-x parts. The two sides of the resultant equations must still be equal after this subdivision; that is, the x part and the not-x part. This follows from the fact that our so-called equation is really the identity of a group of individuals when indicated under two different names or groups of names. Hence the same identity persists when the group is split into two distinct sections, so that the x-parts of the equation and the not-x parts, taken separately, will each still hold good.

We may therefore substitute for the single equation $w = w\,(xy + \bar{x}z)$, the pair

$$\begin{cases} wx = wxy \\ w\bar{x} = w\bar{x}z. \end{cases}$$

[1] For variety, and to remind the reader of their exact equivalence, I occasionally adopt the form $w = wX$ instead of $w = \frac{0}{0}X$.

And for the single one, $w = w(xyz + \bar{x}\bar{z})$, the pair

$$\begin{cases} wx = wxyz \\ w\bar{x} = w\bar{x}\bar{z}. \end{cases}$$

The combined effect of these four being precisely equivalent in every way to that of the original two, or to the one into which we saw we could compress them.

If we had chosen to make a somewhat different arrangement we might have broken up the two sides in respect of y instead of in respect of x. In that case, as the reader will readily see, we should have got the following group instead;—

$$\begin{cases} wy = wy(x + \bar{x}z) \\ w\bar{y} = w\bar{y}\bar{x}z. \end{cases}$$

$$\begin{cases} wy = wy(xz + \bar{x}\bar{z}) \\ w\bar{y} = w\bar{y}\bar{x}\bar{z}. \end{cases}$$

(Attention may be directed here to the second and fourth of these equations. They refer $w\bar{y}$ to two contradictory classes, viz. to one that is z and to one that is not-z. This is, as we know, their way of reminding us that there *is* no such class as $w\bar{y}$; i.e. of saying that 'all w is y'.)

The following remarks apply to the strict Boolian, or exclusive system of notation.

From what has been said it follows that logical equations have some sort of self-righting power about them. In this they are somewhat distinct from mere class expressions, the correction of which, when we have reason to believe them badly expressed, can only be carried out by our guessing what was likely to have been meant. Thus $x + y$, standing by itself, might leave a doubt whether it was meant for 'x or y or both' (i.e. $x + \bar{x}y$) or whether it was merely so put because x and y were known to be independent.

Now compare the equation $x + y = z$.

We see at once that (on the Boolian plan of notation) xy must $= 0$ here, as otherwise a simple class term z would be partly identified with the double term $2xy$. Hence the equation may be written in the strict form, for which it has itself supplied the condition :—

$$x\bar{y} + \bar{x}y = z \text{; (with } xy = 0\text{).}$$

So again in the case of $w(x + y) = x - y$.
When analysed this leads to the denials $wy = 0$, $\overline{w}x\overline{y} = 0$, $\overline{w}\bar{x}y = 0$. Whence we see that y must $= xy$, so that the statement would better have been phrased

$$wx = x\bar{y}.$$

It is then intelligible enough, the only misleading features having been the addition of wy (in that form) to wx, and the apparent introduction of a substantive negative term (y) on one side when both the terms on the opposite side were positive. These difficulties are at once removed by the assurance that there *is* no wy, and that $x - y = x\bar{y}$. What in fact we were originally offered was a piece of somewhat incorrect symbolic grammar; this we are able to correct by the conditions of the system, and so can express the statement in the perfectly unobjectionable form, $wx = x\bar{y}$.

The above remarks become of some symbolic importance in the interpretation of the inverse form $\frac{X}{Y}$. We gave a simple enough explanation of it, applicable to every case in which X and Y were ordinary class expressions. But this explanation will not cover every case. For instance $\frac{x-y}{x+y}$ cannot be read off in any such simple fashion as this. But if we regard it as, so to say, a piece of symbolic bad grammar, it will become amenable to treatment. We know that it might originate

from $w(x+y)=x-y$; that is, 'If w combined with $x+y$ equals $x-y$, what must w be?' Then we have an equation which can be repaired and reconstructed like one of those above considered. For $w(x+y)=x-y$, after reconstruction stands (as shown on the last page)

$$wx = x\bar{y} \text{ with } \bar{x}y = 0.$$

What the expression $w(x+y)=x-y$ asks us, is, to find a class such that on combination with $x+y$ it shall equal $x-y$. It was a foolish question to ask in those words, for we find that the y-part of $w(x+y)$ contributes nothing (wy being $=0$), and that $x-y$ is really $x\bar{y}$, ($\bar{x}y$ being $=0$). So the answer is that $w=x\bar{y}+\frac{0}{0}\bar{x}\bar{y}$.

Boole would not have hesitated to take $\dfrac{x-y}{x+y}$ as it stands, and develope it into $x\bar{y}-\bar{x}y+\frac{0}{0}\bar{x}\bar{y}$. So phrased the expression has no meaning, for we cannot deduct $\bar{x}y$ from either of the other classes. Accordingly he introduces the rule that any term thus affected with a negative factor (or indeed any factor but 1 and $\frac{0}{0}$) must be equated to zero, as we do with those affected with the factor $\frac{1}{0}$. We then find that $\dfrac{x-y}{x+y}=x\bar{y}+\frac{0}{0}\bar{x}\bar{y}$. Take again the following: $\dfrac{1-x-y}{xy}$, which when developed yields, $-xy+\frac{0}{0}x\bar{y}+\frac{0}{0}\bar{x}y+\frac{1}{0}\bar{x}\bar{y}$. This, like the other forms above, will submit to explanation, but it is certainly very badly expressed. Since $1-x-y=\bar{x}\bar{y}-xy$, what we are really asking for, is some factor which will reduce xy to $\bar{x}\bar{y}-xy$. Some class term z is wanted, such that $zxy=\bar{x}\bar{y}-xy$. Of course there can be no such class term, for how can a positive term, like xy, be converted into its contrary $\bar{x}\bar{y}$, or into a negative like $(-xy)$? So the statement escapes nonsense by demanding that xy and $\bar{x}\bar{y}$ shall both $=0$, and then z may naturally be anything what-

ever. The value apparently deduced for z is in fact quite illusory. It stands indeed in the form $\frac{0}{0}x\bar{y}+\frac{0}{0}\bar{x}y$; but, (since $xy=0$ and $\bar{x}\bar{y}=0$,) $x\bar{y}+\bar{x}y=1$, so that $\frac{0}{0}x\bar{y}+\frac{0}{0}\bar{x}y$ is not any determination at all: it is equivalent to $\frac{0}{0}1$, viz. it is absolutely indefinite.

This sort of symbolic solecism, as we may term it, commonly takes the form of asking two questions in one, but asking them in terms which resemble a single question. Thus the expression $\dfrac{0}{x-y}$ yields $\frac{0}{0}xy+\frac{0}{0}\bar{x}\bar{y}$. Now $x-y=x\bar{y}-\bar{x}y$, so that what $\dfrac{0}{x-y}$ asks for is a class such as shall, when combined with $x\bar{y}-\bar{x}y$, reduce to 0. But we cannot, as pretended or suggested, deduct $\bar{x}y$ from $x\bar{y}$. Accordingly the question really means 'Find a class such that combination with *either* $x\bar{y}$ *or* $\bar{x}y$ shall reduce to 0'. This being what we meant, the (symbolically) correct grammatical form would have been $\dfrac{0}{x\bar{y}+\bar{x}y}$. This would develope at once into $\frac{0}{0}xy+\frac{0}{0}\bar{x}\bar{y}$.

It may be noticed that these awkward expressions will often meet us in the solution of problems. Thus take the statement $wx=wy$ which in that form is irreproachable. If asked to determine w, we might proceed to say $wx-wy=0$, therefore $w=\dfrac{0}{x-y}$ as above; upon solving which we should obtain $w=\frac{0}{0}xy+\frac{0}{0}\bar{x}\bar{y}$. The awkwardness here of course begins at the step $wx-wy=0$, for we thus apply the form of subtraction or subduction to terms which are not formally entitled to it, though we know that materially, that is, by the conditions of the data, they are so. A less questionable plan would have been to have broken up $wx=wy$ into $wx\bar{y}=0$, $w\bar{x}y=0$; which would lead at once to $w(x\bar{y}+\bar{x}y)=0$ or

$w = \dfrac{0}{x\bar{y} + \bar{x}y}$, a form against which no objection could be raised.

Before coming to the more general question of the full interpretation of our logical equations, attention may be called to a few formulæ of great utility, as appeal to them will solve the large majority of the problems likely to occur in Logic.

What we mostly want in such cases is specific information about some one class term, say x, as yielded by a group of premises. Suppose we have resolved the premises into their constituent denials, the sum-total of these may always be arranged in the form,

$$Ax + B\bar{x} + C = 0 \text{.....................(1)}$$

A, B, C, consisting of combinations of y, z, &c.

The enquiry as to x may take various forms. We may be asked, for instance, for the full class expression of x in terms of A, B, C. If we work this out by Boole's formula, we have, after the slight modifications implied by the attendant conditions,

$$x = B + \tfrac{0}{0}\bar{A}\bar{B}; \text{ with the conditions,}$$
$$AB = 0,\ C = 0 \text{ ....}(\alpha)$$

Or, regarding A, B, as events or marks indicative of x, we might frame the solution in the form of a statement of the conditions under which x does and does not occur. In this case we have the two propositions,

$$\left.\begin{array}{l} Ax = 0: \text{ or, where there is } A \text{ there is no } x \\ B\bar{x} = 0: \text{ or, where there is } B \text{ there is } x \end{array}\right\}\ldots(\beta)$$

Or again, we may find it convenient to throw the answer into the form of a statement as to what follows

from the presence and absence respectively of x. We then have,

$$\left.\begin{aligned}xA = 0;&\ \text{or, where } x \text{ occurs there is no } A\\ \bar{x}B = 0;&\ \text{or, where } x \text{ is absent there is no } B\end{aligned}\right\}\ldots\ldots(\gamma)$$

If to (β) or (γ) we add the condition $C = 0$, then these three solutions are precisely equivalent to each other, and to the original equation (1).

It will be seen that in each case x is determined by A and B only, C not entering into that determination. The full reply to the question what part C plays in the solution will be given in the chapter on Elimination. But it may be pointed out at once that since C is composed of y, z, &c.—terms which also enter into A and B—it is by no means ineffective. We may use the elements into which $C = 0$ may be decomposed, in order to interpret, and probably to simplify, A and B, and consequently to modify x itself.

In the above case we had a result equated to 0, as our starting point. This is the most usual case. But there is another form, which may be called the complementary one, which must now be noticed, where the sum of the terms is equated to 1. This will be familiar to some readers as being equivalent to the starting point taken by Jevons when solving what he terms the *Inverse Problem*[1]. The typical form here is

$$Ax + B\bar{x} + C = 1.$$

We might deduce the value of x directly from this expression, but it is simpler to obtain it as follows. Begin by writing it in the form,

$$(A + C)\,x + (B + C)\,\bar{x} = 1,$$

[1] Discussed further on.

then 'contradict' each side, and we have

$$\bar{A}\bar{C}x + \bar{B}\bar{C}\bar{x} = 0.$$

From this, by the last rule, we obtain the three corresponding formulæ:—

$$x = \bar{B}\bar{C} + \tfrac{0}{0}(AB + C), \text{ with } \bar{A}\bar{B}\bar{C} = 0 \text{ ....}(\alpha)$$

$$\left.\begin{array}{l}\bar{A}\bar{C}x = 0;\ \text{or where } A \text{ and } C \text{ are absent, there is no } x \\ \bar{B}\bar{C}\bar{x} = 0;\ \text{or where } B \text{ and } C \text{ are absent, there is } x\end{array}\right\}(\beta)$$

$$\left.\begin{array}{l}x\bar{A}\bar{C} = 0;\ \text{or, where there is } x \text{ there is } A \text{ or } C \\ \bar{x}\bar{B}\bar{C} = 0;\ \text{or, where } x \text{ is absent there is } B \text{ or } C\end{array}\right\}\text{...}(\gamma)$$

For convenience, a fourth formula may be added which will be justified presently,

$$(x + B + C)(\bar{x} + A + C) = 1 \text{..............}(\delta)$$

It is the exact equivalent of (β) and (γ).

Similar formulæ follow from the explicit introduction of two class terms into our solutions. Instead of merely wanting information about x alone, we may want it about some combination of x and y. In that case we may arrange our group of denials in the four classes yielded by development in respect of these two terms. Our data would then take the form,

$$Axy + Bx\bar{y} + C\bar{x}y + D\bar{x}\bar{y} = 0,$$

A, B, C, D, being combinations of z, w, &c. The corresponding statements to those above, viz. those yielding information as to the occurrence of x and y now take the form

$$\left.\begin{array}{l}Axy = 0:\ \text{or, where there is } A \text{ there is no } xy:\ i.e. \text{ there is} \\ \qquad \bar{x} \text{ or } \bar{y}. \\ Bx\bar{y} = 0:\ \text{or, where there is } B \text{ there is no } x\bar{y}:\ i.e. \text{ there is} \\ \qquad \bar{x} \text{ or } y. \\ C\bar{x}y = 0:\ \text{or, where there is } C \text{ there is no } \bar{x}y:\ i.e. \text{ there is} \\ \qquad x \text{ or } \bar{y}. \\ D\bar{x}\bar{y} = 0:\ \text{or, where there is } D \text{ there is no } \bar{x}\bar{y}:\ i.e. \text{ there is} \\ \qquad x \text{ or } y.\end{array}\right\}$$

$$\left.\begin{array}{llll} xyA = 0; & \text{where there is } xy & \text{there is no} & A. \\ x\bar{y}B = 0; & \quad ,, \quad x\bar{y} & \quad ,, & B. \\ \bar{x}yC = 0; & \quad ,, \quad \bar{x}y & \quad ,, & C. \\ \bar{x}\bar{y}D = 0; & \quad ,, \quad \bar{x}\bar{y} & \quad ,, & D. \end{array}\right\}$$

The complementary expression to the above takes the form,

$$Axy + Bx\bar{y} + C\bar{x}y + D\bar{x}\bar{y} = 1.$$

This can be shown to be equivalent to the expression

$$(\bar{x} + \bar{y} + A)(\bar{x} + y + B)(x + \bar{y} + C)(x + y + D) = 1.$$

Each of these factors itself equals 1, and may therefore be read off as a disjunctive. Briefly stated they stand,

$$\left.\begin{array}{l} \text{If not } A, \text{ then } \bar{x} \text{ or } \bar{y}. \\ \text{If not } B, \text{ then } \bar{x} \text{ or } y. \\ \text{If not } C, \text{ then } x \text{ or } \bar{y}. \\ \text{If not } D, \text{ then } x \text{ or } y. \end{array}\right\}$$

$$\left.\begin{array}{l} \text{If } xy, \text{ then } A. \\ \text{If } x\bar{y}, \text{ then } B. \\ \text{If } \bar{x}y, \text{ then } C. \\ \text{If } \bar{x}\bar{y}, \text{ then } D. \end{array}\right\}$$

These again may be combined. If, for instance, we enquire under what circumstances x or y alone occur, the answer is,

Given $\bar{A}\bar{D}$ we have $x\bar{y}$ or $\bar{x}y$.

Or, if it be asked, what follows on the occurrence of x or y alone, the answer is,

Given $x\bar{y}$ or $\bar{x}y$ then we have B or C.

And so with any other combinations.

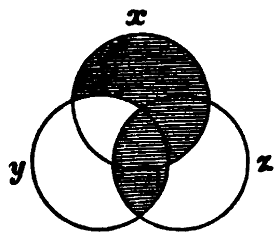
Interpretation of Equations in general.

There are few departments of the Symbolic Logic in which acquired views will have to be more completely abandoned than in reference to the interpretation of our equations. In the common Logic we talk of *the* solution as if there were but one; in fact a plurality of possible answers is considered a fatal defect, so that certain syllogistic figures are rejected on this ground alone.

On the symbolic system all this has, at first sight, to be altered. We must be prepared here for an apparent variety of possible answers. In saying this it is not, of course, implied that conflicting answers could be drawn, but rather that the modes of expression are so various that the same substantial answer can assume a variety of forms.

This distinction rests upon two grounds; firstly, the fact that we put a term and its contradictory (x and $\bar{x}$) on exactly the same footing, whereas the common system seeks always to express itself in positive terms, putting the negation into the predicate. Secondly, there is the obvious difference that whereas but two or three terms are commonly admitted into the former, the latter is prepared to welcome any number.

For instance, take the familiar syllogism, 'all x is y; no z is y; therefore no z is x'. Here it would be said, and very correctly from the customary standpoint, that there is one and only one conclusion possible. Now look at it symbolically: We write the statements in the form $x\bar{y} = 0$, $yz = 0$. Therefore the full combination of the two may be written $x\bar{y} + yz = 0$, and it may be represented in a diagram thus:



It will be seen at once, even in such a simple case, what a variety of possible solutions are here open to us. First take the fully complete solutions. These fall into the usual distinction offered by the positive and negative interpretation; that is, according as we enumerate all the abolished classes, or all the possible surviving ones. Thus $x\bar{y}+yz=0$ expanded into its details gives four terms to be destroyed, the remaining four being equated to unity.

$$\begin{cases} x\bar{y}z + x\bar{y}\bar{z} + xyz + \bar{x}yz = 0 \\ xy\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z} = 1. \end{cases}$$

These are the complete alternative answers given in detail. The former states, with negative disjunction, that there is nothing which falls into any one of certain four classes; the latter, with affirmative disjunction, that everything does fall into one or other of the remaining four classes.

These ultimate elements we may of course group at will, and thus obtain various simplifications of expression. The former we know will stand as $x\bar{y}+yz=0$, the latter will stand as $y\bar{z}+\bar{x}\bar{y}=1$. They then state respectively that there is nothing which is either yz or $x\bar{y}$, and that everything is either $y\bar{z}$ or $\bar{x}\bar{y}$.

These are the complete statements of the information yielded by the data in an implicit form. Now for the same or a part of the same complete information in an explicit form. We may require to have the account of any one of the six terms x, $\bar{x}$, y, $\bar{y}$, z, $\bar{z}$, as described in the other terms; thus

$$x = \tfrac{0}{0}\, y\bar{z}$$

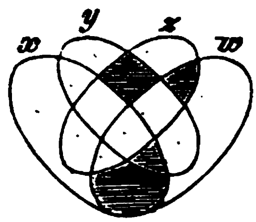
$$\bar{x} = z + \bar{y}\bar{z} + \tfrac{0}{0}\, y\bar{z}$$

$$y = x\bar{z} + \tfrac{0}{0}\, \bar{x}\bar{z}.$$

and so on with the remaining terms.

But even this is only a part of the full problem before us. Our complete scheme comprises two further modifications on anything here indicated. For we may want to determine not merely x and $\bar{x}$, y and $\bar{y}$, z and $\bar{z}$, but any possible combination or function of these; and this we may want to determine not as here in terms of *all* the remaining elements, but in terms of any selection from amongst these, after elimination of the remaining elements. These extensions will be duly discussed in their proper places.

The above example will serve to shew how indefinite is the solution of a logical problem, unless some further indications are given as to the sort of information desired. The sum-total of the facts which are left consistent with the data must necessarily be the same, however they may be expressed. But the various ways of expressing those facts, and still more the various ways of expressing selections and combinations of them, are very numerous. Take, for instance, a slightly more complicated example, such as that indicated in the following figure:



and observe in what a variety of ways the unshaded portion may be described. The figure represents the results of the data;—

$$\begin{cases} \text{All } wx \text{ is } z \\ \text{All } wz \text{ is } x \text{ or } y \\ \text{All } yz \text{ is } w \text{ or } x \end{cases} \quad \text{viz.} \quad \begin{cases} wx\bar{z} = 0 \\ wz\bar{x}\bar{y} = 0 \\ yz\bar{w}\bar{x} = 0. \end{cases}$$

One way of course is to say that the surviving classes are all which are not thus obliterated; these being negatively

$$1 - x\bar{z}w - \bar{x}\bar{y}zw - \bar{x}yz\bar{w}$$

or, slightly grouped,

$$1 - x\bar{z}w - \bar{x}z\,(\bar{y}w + y\bar{w}).$$

Or again, we may express them all positively thus,

$$x\,(wz + \bar{w}z + \bar{w}\bar{z}) + \bar{x}z\,(wy + \bar{w}\bar{y}) + \bar{x}\bar{z}$$

or, less completely positive, but briefer,

$$x\,(1 - w\bar{z}) + \bar{x}z\,(wy + \bar{w}\bar{y}) + \bar{x}\bar{z}$$

or $$x\,(1 - w\bar{z}) + \bar{x}\,\{z\,(wy + \bar{w}\bar{y}) + \bar{z}\},$$

each of these symbolic groupings having of course its suitable verbal description. Thus the last may be read 'All x except what is w but not z; and all not-x, provided it be not-z, or, if z, then both or neither w and y'.

The general nature of the problem thus put before us is easily indicated. Suppose there were four terms involved; then our symbolic apparatus provides 2^4 or 16 compartments or possibilities. The data impose material limits upon these possibilities, leaving only a limited number of actualities. That is, they extinguish a certain number and leave only the remainder surviving. In this case out of the 16 original possibilities 12 are left remaining. The full result then of the data is given by enumerating completely either the extinguished compartments or the remaining ones. Either of these enumerations is only possible in one way, provided we give it in full detail. But when we want to group the results, for more convenience, into compendious statements, we see that this can be done in a variety of ways.

The number of different combinations which can be produced increases with the introduction of every fresh class term in a way which taxes the imagination to follow. Thus three terms yield eight subdivisions. From these we might make eight distinct selections of one only; 28 of a pair; 56 of three together, and so on. The total number of distinct groups which can thus be produced is

$$8 + 28 + 56 + 70 + 56 + 28 + 8 + 1 \text{ or } 255.$$

One case, and only one, is excluded necessarily, namely that in which every compartment is erased, for this corresponds to the one formal impossibility of endeavouring to maintain that every one of our exhaustive divisions is unoccupied: this being, as the reader knows, the symbolic generalization of the Law of Excluded Middle. The arithmetical statement of the total number of cases is readily enough written down. Three terms yield eight sub-classes, viz. 2^3; and these eight sub-classes may be taken as above in $2^8 - 1$ ways: viz. $2^{2^3} - 1$ represents the possible varieties before us. Or expressed generally, if there be n terms we can have 2^n classes, and accordingly $2^{2^n} - 1$ distinct groups of these classes. Of course this expression increases with enormous rapidity as n increases. Four terms thus yield 65,535 possibilities in the way of combination of the elements yielded, and so on.

It may be enquired here if there is no system of classification available for these enormous numbers, so as to make them somewhat less unwieldy to deal with. This is a question which Jevons set himself to answer in what seems to me the most original part of his logical investigations (*Pr. of Science*, p. 137). He describes it as the *Inverse Problem*, and identifies it with the process of Induction.

The following is his table of results for the case of three terms, A, B, C.

Reference Number.	Propositions expressing the general type of the logical conditions.	Number of distinct logical variations	Number of combinations contradicted by each.
I	$A = B$	6	4
II	$A = AB$	12	2
III	$A = B, \quad B = C$	4	6
IV	$A = B, \quad B = BC$	24	5
V	$A = AB, \quad B = BC$	24	4
VI	$A = BC$	24	4
VII	$A = ABC$	24	3
VIII	$AB = ABC$	8	1
IX	$A = AB, \quad aB = aBc$	24	3
X	$A = ABC, \ ab = abC$	8	4
XI	$AB = ABC, \ ab = abc$	4	2
XII	$AB = AC$	12	2
XIII	$A = BC ꖌ Abc$	8	3
XIV	$A = BC ꖌ bc$	2	4
XV	$A = ABC, \ a = Bc ꖌ bC$	8	5

It is simply a classification of the types of proposition where three terms are involved; and is naturally, to a certain extent, arbitrary, in the sense that it is modified by the author's peculiar views as to the nature of propositions, and the relations permissible amongst them. The general meaning is plain enough. Thus VI. indicates that with 3 terms (and their contradictories) in question, we could construct 24 distinct propositions in which a single class is equated to the common part of two others; and that each of these involves the destruction of 4 of the 8 ultimate subdivisions.

For a full explanation of this table and the way to work

it, the reader must be referred to the work in question, but a brief indication may be offered here. Suppose, for instance, we had as the result of a set of premises, the negative conclusions $ABc = 0$, $AbC = 0$[1]. Suppose also that we wished to reduce this to a single proposition, if possible, of the ordinary type. Since two combinations are destroyed, we turn to the last column, and find that there are three types of propositions which produce this exact amount of destruction, viz. II, XI, XII. Accordingly we know that we are to seek amongst these for the desired proposition which is to sum up our result. Of course this is only a portion of our task, for these are *types* merely, containing a variety of species under them. For instance, $A = AB$ is one of twelve distinct species; viz. $A = AB$, $A = Ab$, $a = aB$, $a = ab$, &c.: and similarly with the others. Accordingly every species of proposition of each type may have to be considered before we finally hit upon the right one. The solution we are in quest of, as it happens, is one of those in No. XII, viz. $AB = AC$. That it will produce the required destructions, viz. those of ABc and AbC, and of these only, is evident. We may therefore regard it as the answer.

The nature and use of such a table as this will be readily understood now. Given, as the final outcome of our premises, a certain number of subdivisions destroyed, we want to find one or two propositions which shall sum up, that is, which shall contain within themselves, these destructions. In the case just noticed, the destructions were those of ABc and AbC, and it was found that the one proposition $AB = AC$ would suffice thus to sum them up.

There can be no doubt that much ingenuity and labour

[1] I have employed Jevons' own notation here, as the table is taken from his work. A small letter denotes a negative term; that is, a corresponds to our $\bar{a}$ or $(1 - a)$.

has been devoted to the composition of this table:—in fact it involved, as a preliminary, the discussion of every one of the 255 possible cases which we saw might result from the combinations of three terms. Moreover as a classification of the forms of proposition which would produce these various results it seems to me sound and successful on the whole[1]. But regarded as a means of solving an indeterminate inverse problem, that is of discovering the simplest propositions from which the assigned destructions would result, it does not seem to me equally successful. We must remember that this rather complicated table only includes the consequences of dealing with *three* terms, an unusually small number in even such simple examples as we have employed in this work. With four terms there are 65,535 possible selections of combinations. Jevons considered that it would take several years of continuous labour even to determine the number of possible types of proposition here; and, though Clifford solved this part of the question by showing that there were 396 such types, he still considers that such a period would be required to ascertain what these types actually are. With five terms the number of possible selections is 4,294,967,295, and the number of types is one which presumably no man will ever determine.

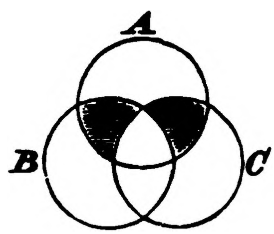
However valuable the direct and indirect results of classifying and analysing these propositional forms may be, any labour bestowed upon them with a view to actually

[1] I differ entirely from some of the assumptions by which the number of admissible combinations is limited. For instance the results are rejected as "inconsistent" whenever they require the simple abolition of any class or its contradictory: e.g. A, B, C, a, b, c. I have already given my reasons for regarding such rejection as arbitrary, and indeed entirely opposed to the spirit of any symbolic method.

There is a good discussion on the general interpretation of a complete scheme of this description, by Miss Ladd, in the Johns Hopkins *Studies*.

solving logical problems seems to me scarcely well bestowed. The requisite investigation appears rather to be of that kind with which tact and judgment, aided by graphical methods, will best grapple. It is naturally allied to that class of physical investigations with which we deal by the graphical method of curves. Suppose a succession of values of some element are given, and we want to determine the law involved in them. What we should do is to draw ordinates corresponding to these values, and trace a curve through their extremities, and then endeavour, by the help of the eye, to detect, at any rate provisionally, what kind of law the curve follows.

In the case of many logical problems there is a corresponding device which we have already had frequent occasion to use, viz. that of diagrams. The diagram corresponding to the two destructions of subdivisions ABc, AbC, is of course:



A glance at such a diagram will show that a good and brief description of the result is that the only A which survives is that which is both B and C and that which is neither. In the notation of Jevons this would be written down $A = A\,(BC + bc)$. At first sight it may seem that this is a different result from that yielded by the table, but a little observation will show that they are equivalent or alternative forms.

Shortly after the above was written Dr Keynes showed (*Formal Logic*, Part IV. Chap. XII.) that more definite help than had formerly been supposed, could be given towards solving this Inverse Problem. I think the problem remains still essentially indeterminate; in the sense that a number of different answers can be given, and that it is difficult to lay down any precise rule as to which of them is to be considered the most correct or simple answer. But methods are available which will enable us to find *an* answer, with tolerable ease. Three such are described by Dr Keynes. The first starts with the affirmative side of the equation, that is, with those combinations which are left undestroyed. If $X + Y + Z + W + \&c. = 1$, we may make a selection from these terms and say, Whatever is neither X nor Z nor ... is Y or W or .... By rules for the contradiction of a group of complex terms the left side of the above statement is broken up into details,—the selection having been purposely made so as to simplify the expression of these,—and we are thus left with a series of comparatively easy disjunctive propositions. The second method deals in a somewhat similar way with the negative side of the original equation, viz. with the group of terms whose destruction was implied by it. The third method appeals to the principle of dichotomy. It begins by selecting, say, the x and $\bar{x}$ constituents from the group of combinations left surviving. If either of these consists of a single element only, we have at once a categorical proposition. If not, we proceed to select, in the same way, the y and $\bar{y}$ parts, and so on. The result is a series of categorical propositions with simple or complex subjects and predicates.

This last method was afterwards modified by Mr W. E. Johnson (*Mind*, Vol. I. New Series), who by the employment of a novel scheme of arrangement has produced a very

elegant and effective mode of dealing with this Inverse Problem. To understand it the reader must recall the formulæ given on pp. 313—6. It was pointed out that if we have a series of factors equated to unity, $XYZ\ldots=1$, it follows that each of these factors is separately $=1$. It was also shown that $(ac+\bar{a}d)=(a+d)(\bar{a}+c)$; what we thus do being to break up a sum of products into a product of sums.

Here comes in the advantage of arrangement, as will soon be manifest. Write the latter formula in the following way,

$$\begin{array}{c|c} a & c \\ \hline d & \bar{a} \end{array}$$

where terms on the same horizontal lines are to be understood as connected by multiplication, and those on the same vertical line by addition. Now suppose that c and d are themselves complex, and proceed to resolve and arrange them in the same way with respect to another class term, say y; and proceed in this way until every sum has been resolved into products. If we then read the whole expression in horizontal lines taken successively downwards, we have the original expression: if we read it in vertical lines taken successively sideways we have the sort of expression we want. The main use of the peculiar arrangement consists in the help which the eye affords in so grouping the terms as to obtain the simplest 'product of sums' for our special purpose.

Thus, suppose we have

$$xyz\bar{w}+x\bar{y}\bar{z}w+\bar{x}yzw+\bar{x}\bar{y}\bar{z}\bar{w}=1.$$

What the inverse problem proposes is to reduce this to an equivalent in the shape of a number of ordinary propositions. Arrange it, as above described, selecting first the x and $\bar{x}$

terms, and then selecting from each of these the y and $\bar{y}$ terms. We have

$$\begin{array}{cc|c|c} \multicolumn{2}{c|}{} & y & z\bar{w} \\ \cline{3-4} \multicolumn{2}{c|}{x} & \bar{z}w & \bar{y} \\ \hline y & \multicolumn{1}{|c|}{zw} & \multicolumn{2}{c}{} \\ \cline{1-2} \bar{z}\bar{w} & \multicolumn{1}{|c|}{\bar{y}} & \multicolumn{2}{c}{\bar{x}} \end{array}$$

If we now read this off in successive horizontal lines, and in successive vertical lines, regarding x and $\bar{x}$ as if they were bracketed with respect to the groups beside and opposite them, we have ;—

$$xyz\bar{w} + x\bar{y}\bar{z}w + \bar{x}yzw + \bar{x}\bar{y}\,\bar{z}\bar{w} =$$

$$(x + y + \bar{z}\bar{w})\,(x + \bar{y} + zw)\,(\bar{x} + y + \bar{z}w)\,(\bar{x} + \bar{y} + z\bar{w}).$$

Since each of these totals $= 1$, we know that all the factors of the second separately $= 1$. But the expression

$$x + y + \bar{z}\bar{w} = 1,$$

interpreted as a disjunctive, says that 'what is not (x or y) is $\bar{z}\bar{w}$'; i.e., All $\bar{x}\bar{y}$ is $\bar{z}\bar{w}$. Similarly with the other factors. Hence the original expression is resolved into the four propositions,

All $\bar{x}\bar{y}$ is $\bar{z}\bar{w}$,

All $\bar{x}y$ is zw,

All $x\bar{y}$ is $\bar{z}w$,

All xy is $z\bar{w}$.

This then is *a* solution of the problem proposed; but there is no particular reason for choosing it in exclusion of other equivalent solutions, except on the score of its com-

parative simplicity. It should be observed that some tact and experience are needed in order to secure this simplicity. Assuming, for instance, that we want only categorical propositions in our solution, it would not do, in the above case, to admit more than one complex term into any one of the factors; as we should not then obtain such a single proposition as we want. Thus $x + y + \bar{z}\bar{w} = 1$, gives at once, All $\bar{x}\bar{y}$ is $\bar{z}\bar{w}$. But $xy + \bar{z}\bar{w} = 1$, though symbolically just as simple, when converted into a disjunctive reads, All that is not xy is $\bar{z}\bar{w}$; which yields the *two* simple categoricals; All $\bar{x}$ is $\bar{z}\bar{w}$, All $\bar{y}$ is $\bar{z}\bar{w}$. The great merit of Mr Johnson's scheme is that a glance of the eye will serve to secure the most desirable arrangement for the purpose in view.

It may be pointed out that diagrams are very serviceable for *testing* our results in this case. In the case above, for instance, draw the four-ellipse figure; and, beginning with the original equation mark off with (say) a small vertical line all the component subdivisions included in it. Then take the four propositions in the solution, resolve each into its elementary denials, and mark off all the subdivisions, thus accounted for, by a small horizontal line. If the solution is correct every subdivision in the diagram must be marked with one of the two kinds of line, but none with both. All that this means is, of course, that every one of the 16 possible combinations of four terms must be accounted for in the way either of denial or of non-denial. Where more than four or five terms are involved the figures described on page 140 will be found very convenient for this purpose.

Another method of breaking up an expression into its ultimate factors is by double negation (see Schröder, I. 357). Suppose, for instance, we have the expression

$$xy + xz + xw + yzw + u.$$

Call this S. Then

$$\bar{S} = (\bar{x} + \bar{y})(\bar{x} + \bar{z})(\bar{x} + \bar{w})(\bar{y} + \bar{z} + \bar{w})\bar{u}.$$

Multiply this out, and we have

$$\bar{x}\bar{y}\bar{u} + \bar{x}\bar{z}\bar{u} + \bar{x}\bar{w}\bar{u} + \bar{y}\bar{z}\bar{w}\bar{u}.$$

Contradict this again, and we revert to S, or the original expression, which therefore becomes

$$(x + y + u)(x + z + u)(x + w + u)(y + z + w + u).$$

If this is multiplied out we are brought back again to the original statement.

CHAPTER XIII.

MISCELLANEOUS EXAMPLES.

THIS chapter is entirely devoted to the discussion of examples. In so abstract a subject as this it is not easy to explain principles, at any rate to beginners, except by aid of a variety of concrete instances. This will be a justification for treating the first few examples at considerable length. Had our object been to show the possible brevity of the methods at our disposal most of these examples could have been worked out in a few lines.

(1) Suppose we were asked to discuss the following set of club rules, in respect of their mutual consistency and brevity[1]:—

α. The Financial Committee shall be chosen from amongst the General Committee.

β. No one shall be a member both of the General and Library Committees unless he be also on the Financial Committee.

γ. No member of the Library Committee shall be on the Financial Committee.

[1] For references to other solutions see Schröder, I. 541.

Put x to stand for the class constituting the Financial Committee, y for that of the Library, and z for that of the General. Then the three given rules will stand as follows:—I state them, both in the form most natural to the ordinary Logic, and in the equivalent negative form which is most convenient for our symbolic system.

$$\left.\begin{array}{ll}(\alpha) & \text{All } x \text{ is } z \\ (\beta) & \text{All } yz \text{ is } x \\ (\gamma) & \text{No } x \text{ is } y\end{array}\right\}. \qquad \left.\begin{array}{r}x\bar{z}=0 \\ \bar{x}yz=0 \\ xy=0\end{array}\right\}.$$

As the rules originally stood, they might very likely pass muster. But develop the symbolic expressions by our process of subdivision, or rather develop in this way rules (α) and (γ), for rule (β) already stands in its lowest terms.

They then stand thus:—

$$\left.\begin{array}{r}xy\bar{z}+x\bar{y}\bar{z}=0 \\ \bar{x}yz=0 \\ xyz+xy\bar{z}=0\end{array}\right\}.$$

It is now obvious that the third rule is partially redundant, or at any rate that the first and third together are so; for they overlap each other by both denying the same element $xy\bar{z}$. Accordingly they would stand better stated without redundancy in the form,

$$\left.\begin{array}{r}xy\bar{z}+x\bar{y}\bar{z}=0 \\ \bar{x}yz=0 \\ xyz=0\end{array}\right\}.$$

Of course when we have thus analyzed and corrected them there are reasons of brevity for seeking to express them in the most compendious form. The best way of effecting this (which it must be observed is a matter of tact and skill, for which no strict rules can be given) is to put the first rule back into its original shape, and then to

combine the other two into one. As regards the last step: symbolically, we just add the two together; logically, we say that if there can be no yz which is x, and none which is not, then there can be no yz at all. The whole force of the rules therefore is fully expressed by the two equations:—

$$\left.\begin{array}{r} x\bar{z} = 0 \\ yz = 0 \end{array}\right\}.$$

Or in words,

All the Financial Committee are on the General Committee.
None of the Library Committee are on the General Committee.

Simple as it is, such an example brings out clearly one of the uses of the process of Development or Subdivision. When we proceed to break up the separate assertions into their ultimate details, it becomes easy to see whether or not they partially overlap; and if so, whether these overlapping parts are, as is most often the case, simple redundancies, or whether they amount to direct inconsistencies. Such redundancies and inconsistencies would in general be obvious enough at once when they are complete; but when they are only partial, especially when they are expressed in the vague forms of ordinary language, they may easily escape notice.

The symbolic statements are not strictly inconsistent, though they are to some extent redundant. But when we look at the verbal statements we find that these convey misleading suggestions. "No one shall be a member of the Library and General Committees, unless he be also on the Financial":—this does not, it is true, assert that there is such a class of persons as those who form the subject of this proposition, but it would always be understood most

strongly to suggest that there is. The symbolic statement is quite clear upon this point. It confines itself to denials only, and to these it unconditionally adheres.

(2) Three persons A, B, C, are set to sort a heap of books in a library. A is told to collect all the English historical works, and the bound foreign novels: B is to take bound historical works, and the English novels, provided they are not historical: to C is assigned the bound English works and the unbound historical novels. What works will be claimed by two of them? Will any be claimed by all three?

Put a for	English,	then	$1-a$[1]	stands	for	foreign.
b „	historical,	„	$1-b$	„	„	not-historical.
c „	bound,	„	$1-c$	„	„	unbound.
d „	novels,	„	$1-d$	„	„	not-novels.

The propositions therefore assigning the apportionment of the various books will then stand as follows; A standing for the books assigned to A and so on;—

$$\left.\begin{aligned} A &= ab + (1-a)\,cd \\ B &= bc + (1-b)\,ad \\ C &= ac + (1-c)\,bd \end{aligned}\right\}.$$

Now we know that x and y being any class terms, the expression xy stands for what is common to both x and y. Hence to indicate generally the books belonging to A and B, we should write AB; or if, as here, we want to know the classes in their details, we should multiply together the detailed descriptions of them; that is,

$$\{ab + (1-a)\,cd\}\,\{bc + (1-b)\,ad\}.$$

[1] $\bar{a}$ is the abbreviated form, of course; but I purposely choose various forms of expression here. For reference to other solutions of this example see Schröder, I. 392.

'Multiply' out this product, then, according to the symbolic rules of logic, and we have

$$AB = abc + (1 - a)\, bcd,$$
$$= bc\, \{a + (1 - a)\, d\}.$$

That is, put into words, we have assigned to both A and B the class of books describable as 'bound historical works; whether English generally, or foreign novels.' Precisely similar expressions would be yielded for BC, and AC, viz. for the books assigned to B and C, and to A and C.

In the same way if the question be to find what books are assigned to all three persons, we should have to find the details of the expression ABC. This is simpler in its result, since more of the terms neutralize each other; we find in fact

$$ABC = abc.$$

That is, the only books which have been thus assigned to all three are the 'bound English historical works'.

(3) The following may be offered as an example of the inverse process corresponding to the sign of division;—It is found that when all the books in a library, except philosophy and divinity, are rejected, they are reduced to philosophy and protestant divinity; but include all the works on those subjects. What is the widest and the narrowest extent, so far as expressible in these class terms, which the library could have possessed under the given conditions?

What is really asserted here, is, that when the restriction of being confined to 'philosophy and divinity' is imposed upon the books, they are exactly reduced to 'philosophy and protestant divinity'.

Put x for philosophy,
y „ divinity,
z „ protestant,

and let w represent the class, whatever it may be, constituting the library:

Then the data assert, when expressed in the strictest symbolic form,

$$(x+\bar{x}y)\,w = x + \bar{x}yz.$$

The combination of $(x+\bar{x}y)$ with w, by the sign of multiplication, indicates the restriction of w by the stated condition, and this is to be equated to $x+\bar{x}yz$. The inverse problem therefore is, Find w.

Hence $$w = \frac{x+\bar{x}yz}{x+\bar{x}y}.$$

Develop this expression in accordance with the rules, and we obtain,

$$w = x + \bar{x}yz + \tfrac{0}{0}\,\bar{x}\bar{y}.$$

That is, the library must have certainly contained all philosophy and protestant divinity, and may possibly have contained any kind of works which are neither philosophy nor divinity: this latter constituent being left entirely indefinite.

The two following are miscellaneous examples of this inverse process:—

(4) There is a certain class of things from which A picks out the 'x that is z, and the y that is not z', and B picks out from the remainder 'the z which is y and the x that is not y'. It is then found that what is left exactly comprises the class 'z which is not x'. What can be determined about the class originally?

Call that class w. Then the statement amounts to this, symbolically;—

$$w\,(1 - xz - y\bar{z})\,(1 - yz - x\bar{y}) = \bar{x}z.$$

For A, picking out the $(xz + y\bar{z})$ part, reduces it from w to $w - w(xz + y\bar{z})$, or converts it into $w(1 - xz - y\bar{z})$; similarly B's selection has the same effect of reducing this by the multiplication of $(1 - yz - x\bar{y})$. This final result is then declared to be exactly equivalent to $\bar{x}z$, as above expressed.

On multiplying out, most of the terms in the brackets disappear, and we have

$$w\bar{x}\bar{y} = \bar{x}z$$

or

$$w = \frac{\bar{x}z}{\bar{x}\bar{y}},$$

$$\therefore w = \bar{x}\bar{y}z + \tfrac{0}{0}(1 - \bar{x}\bar{y})(1 - \bar{x}z),$$

$$= \bar{x}\bar{y}z + \tfrac{0}{0}(x + \bar{x}y\bar{z}) \quad\ldots\ldots\ldots(1),$$

with the condition

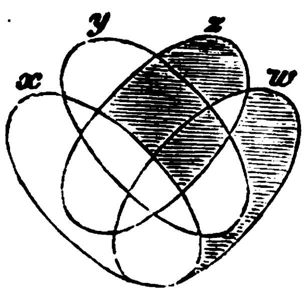
$$\bar{x}z(1 - \bar{x}\bar{y}) = 0,$$

or

$$\bar{x}yz = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

That is, the class must have certainly consisted of 'all z that is neither x nor y' and may have also contained 'anything that is x, or that is y but neither x nor z', but nothing else. Moreover by the terms of statement 'all yz is x'.

I append the diagram, as it may serve to aid conviction.



The reader will readily see that if from w as thus composed we make the two assigned selections, there will be left 'z that is not x', and that only. This is equally so in whichever order we make the two selections.

(5) Of the candidates for a certain examination it was found that the plucked exactly comprised the boys who took Greek and the girls who took Latin. Find the full description of the boy candidates as describable in the other terms thus introduced.

Put x for Latin.
z „ Greek.
w „ boys, $\therefore\ \overline{w}$ = girls (these being contradictories in that universe).
y „ plucked.

The statement is that,

$$y = x\overline{w} + zw,$$

$$\therefore\ w(z - x) = y - x,$$

$$\therefore\ w = \frac{y - x}{z - x}.$$

The numerator and denominator not being here, separately, as they stand, interpretable class expressions, we may adopt the full formula

$$f(x, y, z) = f(1, 1, 1)\, xyz + \&c.$$

Hence $$w = x\overline{y}\overline{z} + \overline{x}yz + \tfrac{0}{0}(xyz + \overline{x}\overline{y}\overline{z})\ldots\ldots\ldots\ldots(1),$$

with conditions $$x\overline{y}z + \overline{x}y\overline{z} = 0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

That is, in words, The boys comprise

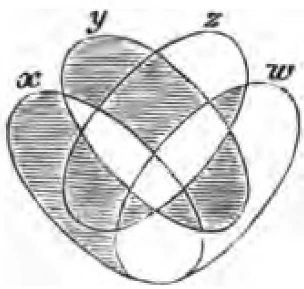
the whole of
- those who take Latin but not Greek, and were not plucked.
- „ „ „ Greek but not Latin, and were plucked.

an uncertain portion of
- „ „ „ Greek and Latin, and were plucked.
- „ „ „ Neither Greek nor Latin, and were not plucked.

With the implied conditions

$\left\{\begin{array}{l}\text{All who took both Latin and Greek were plucked.}\\ \text{All who were plucked took either Latin or Greek.}\end{array}\right.$

In practice the resort to $f(x, y, z)$ is needlessly elaborate. Had we phrased the expression in mutually exclusive terms, $w = \dfrac{\bar{x}y - x\bar{y}}{\bar{x}z - x\bar{z}}$, we might have expanded at once in accordance with the rule for the simplest form of fraction. We may get the same result by more logical considerations. The expression $w(\bar{x}z - x\bar{z}) = \bar{x}y - x\bar{y}$ equates the differences between two mutually exclusive terms. This can only be rendered intelligible (in accordance with the discussions in the next chapter) by separately equating the positive and negative parts; which yields $w = \dfrac{\bar{x}y}{\bar{x}z}$, $w = \dfrac{x\bar{y}}{x\bar{z}}$. Hence we get, $w = \bar{x}y + \frac{0}{0}(1 - \bar{x}z)$, $w = x\bar{y} + \frac{0}{0}(1 - x\bar{z})$. The combined effect of these two determinations of w makes it include all $\bar{x}y + x\bar{y}$, and an uncertain part of $xz + \bar{x}\bar{z}$. This coincides with (1), regard being had to the conditions of (2).



A few moments' attention may be directed here to the diagrammatic interpretation of the symbol $\frac{0}{0}$. When we look at the four surviving classes which compose the figure w, it might be thought that they all stood upon an equal footing of certainty, and that there was nothing in them

corresponding to the sharp distinction represented symbolically by

$$w = x\bar{y}\bar{z} + \bar{x}yz + \tfrac{0}{0}\,xyz + \tfrac{0}{0}\,\bar{x}\bar{y}\bar{z}.$$

But reflexion soon shows the difference. The two former are completely describable in terms of x, y, and z. The two latter cannot be so described without making use of w also. But w being the very term to be defined, the admission of it into the definition is equivalent to no determination at all. We may make the semblance of a determination by writing it

$$w = x\bar{y}\bar{z} + \bar{x}yz + wxyz + w\bar{x}\bar{y}\bar{z},$$

but such avoidance of indeterminateness is altogether delusive. This class w therefore contains the whole of $x\bar{y}\bar{z}$ and $\bar{x}yz$, and 'a part' of xyz and $\bar{x}\bar{y}\bar{z}$. Whether this 'part', or 'some', will really prove to be the whole, or a part only, or even none, will depend upon circumstances. If the given premisses are to be regarded as final, and as postulating the existence of all which they do not deny, then this 'part' is really a part only, and the word *some* in its common signification might be substituted for $\tfrac{0}{0}$.

(6) As an instance of the implied extinction of a whole class, take the following:—

At a certain town where an examination is held, it is known that,

1. Every candidate is either a junior who does not take Latin, or a senior who takes Composition.
2. Every junior candidate takes either Latin or Composition.
3. All candidates who take Composition, also take Latin, and are juniors.

Show that if this be so there can be no candidates there.

Putting x for candidates, a and $\bar{a}$ for junior and senior, c for those who take Latin, and e for those who take Composition, the data stand thus:—

$$\begin{cases} x = \frac{0}{0}(a\bar{c} + \bar{a}e), \\ ax = \frac{0}{0}(c + \bar{c}e), \\ ex = \frac{0}{0}\, ac. \end{cases}$$

These resolve at once into the negations

$$\left.\begin{matrix} x\bar{a}\bar{e},\ xac,\ xc\bar{e} \\ ax\bar{c}\bar{e} \\ \bar{a}ex,\ \bar{c}ex \end{matrix}\right\} = 0.$$

Gathering together the factors of x we have,

$$x\,\{\bar{a}\bar{e} + \bar{a}e + a\,(c + \bar{c}\bar{e}) + c\bar{e} + \bar{c}e\} = 0.$$

This accounts for *all* the possible factors of x; as would be seen at once by subdividing these terms, when all the eight possible combinations of a, c, e, will be found to be represented here. In other words, there is no x.

(7) In the last example one whole class was seen to have perished. In such cases, or when the destruction is even more extensive, the results are often more easily interpreted by aid of the diagrams than by relying only on symbols. For instance,

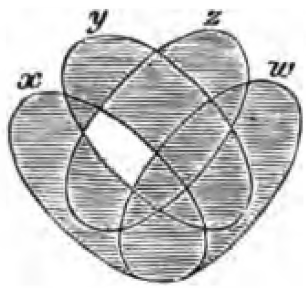
$$\begin{cases} x = y + z\bar{y}, \\ y = \bar{z} + z\bar{w}, \\ \bar{z}\bar{w} = 0, \\ xw = yzw. \end{cases}$$

Looking out for the appropriate multipliers, as already indicated, the following classes are abolished:—

(1) $x\bar{y}\bar{z}$, $\bar{x}y$, $\bar{x}\bar{y}z$. (2) yzw, $\bar{y}\bar{z}$, $\bar{y}z\bar{w}$.

(3) $\bar{z}\bar{w}$. (4) $x\bar{y}w$, $x\bar{z}w$, $\bar{x}yzw$.

Shade out in the diagram, and it stands thus :—



(The outside, or $\bar{x}\bar{y}\bar{z}\bar{w}$ should also be shaded, to make the figure a complete representation of the data.)

It is obvious at a glance that all w is destroyed, and that the only surviving compartment is xyz. This would be described verbally in the statements,

$$\begin{cases}\text{Everything is } x, y, \text{ and } z.\\ \text{There is no } w.\end{cases}$$

(The same example, with other letters, was discussed in the note on p. 164.)

The next two examples have some historic interest, as representing early attempts at the symbolic solutions of problems :—the first of them is almost intuitively obvious.

(8) If x that is not a is the same as b, and a that is not x is the same as c, what is x in terms of a, b, and c? (Lambert, *Log. Abh.* I. 14.)

Since $\qquad x = ax + \bar{a}x, \therefore x - \bar{a}x = ax, \quad \text{or } x - b = ax.$

Since $\qquad a = ax + a\bar{x}, \therefore ax = a - a\bar{x} = a - c.$

$$\therefore x - b = a - c, \text{ or } x = a + b - c.$$

(It will be seen, from the terms of statement, that a and b are mutually exclusive; also that c is a part of a. Hence the whole result, as it stands, is really correctly stated, though it apparently involves double counting and

inappropriate subtraction.) I append Lambert's solution in his own notation;—

"Man habe $x|a=b$; $a|x=c$, so ist [by a former result] $ax=a-c=x-b$; Folglich $x=a+b-c$."

(9) If $xy=zw$, is it correct to conclude that $\frac{x}{z}=\frac{w}{y}$? Or, put into words, if the members common to x and y are the same as those common to z and w, does it follow that the class which on restriction by z will reduce to x is the same as that which on restriction by y will reduce to w? Lambert more than once assumes that this is so, but it is soon seen (I think) that he was in error.

The way in which I am inclined to regard the question is as follows. The expression $xy=zw$ is plain enough. It demands, and is satisfied by, the abolition of all the constituent elements of the two equated classes except the one common element $xyzw$: that is, it involves the denial of $xy\bar{z}$, $xy\bar{w}$, $\bar{x}zw$, $\bar{y}zw$.

Now $\frac{x}{z}=\frac{w}{y}$, when developed, yields $xz+\frac{0}{0}\bar{x}\bar{z}=wy+\frac{0}{0}\bar{w}\bar{y}$. The elementary denials involved in the latter equation (see p. 301) are those of $x\bar{y}zw$, $xyz\bar{w}$, $\bar{x}yzw$, $xy\bar{z}w$: which are a portion only of the combinations assigned above. Moreover, as we have throughout interpreted our fractional forms, each of the equated expressions involves a further condition; viz. we have also $x\bar{z}=0$, $w\bar{y}=0$. These contain *more* than the terms omitted above[1].

[1] My conclusion in the first edition was disputed by Dr A. Macfarlane (*Phil. Mag.* July, 1881), and by Mr H. H. Turner, of Trinity College, Cambridge (now Professor of Astronomy at Oxford), who both maintained that Lambert was right. I have given the subject the careful reconsideration which criticism from such sources demanded, but am unable to agree with their view; though I have somewhat altered my own explanation. As I afterwards saw, and as Prof. Schröder has pointed out, there was an error in the former explanation.

No doubt the four following steps seem to follow naturally: $xy = zw$, $\therefore x = \frac{zw}{y}$, $\therefore x = z \cdot \frac{w}{y}$, $\therefore \frac{x}{z} = \frac{w}{y}$. If the reader works out, in each case, the constituent denials (the best test, I conceive) he will find that the first two and the last two are identical in their implications, the faulty step being that from the second to the third. That is, the associative law does not hold good as regards fractions: we cannot step from $\frac{zw}{y}$ to $z \cdot \frac{w}{y}$. It is not true that 'the class which on restriction by y will reduce to wz is the same as the z part of the class which on restriction by y will reduce to w.'

It deserves notice that the converse step, viz. that from $\frac{x}{z} = \frac{w}{y}$ to $xy = zw$, does seem to hold good. I do not mean that we are warranted in multiplying up, as with arithmetical fractions, but that when we examine in detail the ultimate denials involved in the two expressions, we find that those of $xy = zw$ are included as a part of those yielded by $\frac{x}{z} = \frac{w}{y}$.

Lambert's error deserves the more notice, because he had gained a remarkable grasp of the truth in an analogous, though somewhat simpler case. He distinctly states that we cannot put logical division on the same footing as ordinary division, by striking out common factors: and this, not because the result would be false, but because it would not be general enough. That is, we ought to have an indeterminate result, instead of a determinate one. His words are,......"Wenn $x\gamma = \alpha\gamma$, so ist $x = \alpha\gamma\gamma^{-1} = \alpha\frac{\gamma}{\gamma}$. Aber deswegen nicht allezeit $x = \alpha$; sondern nur in einem einzigen Falle, weil x und α zwei verschiedene Arten von dem Geschlecht $x\gamma$ oder $\alpha\gamma$ sein können. Wenn aber $x\gamma = \alpha\gamma$

nicht weiter bestimmt wird, so kann man unter andern auch $x = a$ setzen." That is, $x = a$ is *one* solution of the problem.

(10) The following example shows the simplification which may be effected by a suitable choice of the 'universe of discourse.'

There are four girls at school, Anna, Bertha, Cora and Dora: it has been observed that

(1) When Anna or Bertha (or both) remained at home, Cora was at home.

(2) When Bertha was out, Anna was out.

(3) When Cora was at home Anna was at home.

What information is here conveyed concerning Dora? (Dr Marquand, *Proc. of Am. Acad. of Arts and Sciences*, Vol. XXI.)

At first sight this seems to involve five terms. But we soon see that Dora, not being introduced into the premises, stands on the same footing as any other object in existence; also, that as no other contingencies are contemplated than that of a girl being in or out, we may restrict our universe to these[1], and accordingly designate the circumstance of Anna being in and out by the symbols a and $\bar{a}$. We then want only three terms for the premises, which become

$$\begin{cases} (a+b)\bar{c} = 0 \\ \bar{b}a = 0 \\ c\bar{a} = 0. \end{cases}$$

The result is to destroy all combinations except abc and $\bar{a}\bar{b}\bar{c}$: or, in words, the three first girls always stay together,

[1] It must be remarked that this interpretation confines our symbolic apparatus to a 1 and 0 system. If a girl is not in she must be out: there is no intermediate case here corresponding to the notation for particular propositions. This interpretation will recur for discussion in Chap. XVIII.

whether at home or out. Accordingly Dora, like any other companion, must stop at home with them all or go out with them all.

A glance at the symbolic statement of the premises will show that they are the three contradictions involved in the equation

$$c = a = b,$$

which will therefore take their place. This would be expressible in the words, The occurrence of c is exactly coextensive with the occurrences of a and of b.

(11) If a genus A is divisible into the species $x, y, z, \ldots$ and also into $\alpha, \beta, \gamma, \ldots$; and if we know that all x is α, all y is β, all z is γ, and so on; then, conversely, all α is x, all β is y, all γ is z, and so on.

This example is taken from Hauber[1], and can be solved at once under the conditions implied. The species he understands to be mutually exclusive, so that we may write the data,

$$1 = \alpha + \beta + \gamma + \ldots \nu$$
$$1 = x + y + z + \ldots n.$$

Now we are told that the species are mutually exclusive; also that x is included in α, y in β, γ in $z, \ldots \nu$ in n. We may therefore (in accordance with the conditions described when we discussed the nature of logical subtraction) subduct the latter from the former, when we have

$$0 = (\alpha - x) + (\beta - y) + \ldots + (\nu - n).$$

But, since x is included in α, $\alpha - x = \alpha\bar{x}$, and similarly with the other terms;

$$\therefore\ 0 = \alpha\bar{x} + \beta\bar{y} + \gamma\bar{z} + \ldots + \nu\bar{n},$$

that is, 'all α is x', 'all β is y', 'all γ is z', etc.

[1] *Scholæ logico-mathematicæ*, 1829. A work mainly mathematical, but containing several good logical hints and generalizations.

The above solution (very slightly altered in form here) was objected to by Prof. Schröder (*Vorlesungen*, II.) as involving illicit mathematical subtraction. After reconsideration, I cannot but maintain that it is quite justifiable logically. Take a material example, which is sufficiently true for mere illustration. Suppose I say that the University consists of graduates and undergraduates, and also of those who have a voice in the management and those who are *in statu pupillari;* and let it be known that all graduates have a voice and that all undergraduates are *in statu pupillari.* Then it seems to me to be a valid process of logical subduction or exception, as distinguished from mathematical subtraction, to say that, if we take the former classes respectively from the latter, there is nothing left; and therefore to reason as above.

(12) Much of the prejudice which for some time existed against the employment of symbolic methods in Logic must be attributed, unless I am mistaken, to the extreme length and elaboration with which Boole generally worked out his results[1]. What he seemed to care for was theoretic perfection of method in the generality of his solution, and consequent certainty of obtaining the desired result, rather than practical convenience in working. It may be well therefore to point out in how many cases this tediousness of operation may be avoided. Here, as in mathematics, there are a variety of devices open to us by which particular conclusions may often be easily obtained.

Take, for instance, the following example from Boole (*Laws of Thought*, p. 118,—the wording is slightly altered).

[1] Much of the progress which has been recently made in the subject consists in the invention of intermediate formulæ, similar to those with which the reader will be familiar in Algebra and Trigonometry. Direct appeal to such formulæ produces much economy in practical work.

(1) Wherever the properties x and y are combined, either the property z or the property w is present also, but not both of them.

(2) Wherever y and z are combined, x and w are either both present or both absent.

(3) Wherever x and y are both absent, z and w are both absent also, and vice versâ.

The problem then is, to determine what may be inferred from the presence of x with respect to the presence or absence of y and z, paying no regard to w; that is, eliminating w.

The symbolic statement of these data stands as follows:—

$$\begin{cases} xy = \frac{0}{0}(w\bar{z} + \bar{w}z) \\ yz = \frac{0}{0}(xw + \bar{x}\bar{w}) \\ \bar{x}\bar{y} = \bar{w}\bar{z}. \end{cases}$$

Now it is clearly no good doing anything which will make the left side of the first two equations vanish; for this, leaving only indeterminate terms, will lead to nothing. Accordingly we need only ask what factors will make the right side disappear. As regards the first equation, either $\bar{w}$ or z will destroy the first term, and either w or $\bar{z}$ will destroy the second. Of the four combinations thus produced two, viz. $w\bar{w}$ and $z\bar{z}$, are of course ineffective, since they destroy both sides. This leaves two available factors, viz. wz and $\bar{w}\bar{z}$. Thus the first equation breaks up into $xyzw = 0$, $xy\bar{z}\bar{w} = 0$. Similarly for the second there are two available factors, viz. $x\bar{w}$ and $\bar{x}w$, the other two being ineffective, as in the last case. Consequently this yields $xyz\bar{w} = 0$, $\bar{x}yzw = 0$. In the third case there being no indeterminate factor, the abolition of either term will lead to something. Now either w or z will destroy the right side, and either x or y the left. Consequently, by multiplication, we get four denials here.

The net results of the three premises may be thus written down:—

$$\left.\begin{array}{ll} xywz & x\bar{w}\bar{z} \\ xy\bar{w}\bar{z} & y\bar{w}\bar{z} \\ xy\bar{w}z & \bar{x}\bar{y}w \\ \bar{x}ywz & \bar{x}\bar{y}z \end{array}\right\} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

A very little practice and experience would of course avoid even this amount of trouble, for 'All A is either B or C only' is equivalent to 'There is no A which is both B and C, and there is none which is neither'. That is, we should multiply the left term by the *full* contradiction of the right term, instead of taking the elements of this contradiction in detail. The full contradiction of $w\bar{z}+\bar{w}z$ is $wz+\bar{w}\bar{z}$, and that of $xw+\bar{x}\bar{w}$ is $x\bar{w}+\bar{x}w$. The first column in (1) might therefore have been written down at once

$$xy\,(wz+\bar{w}\bar{z})=0, \qquad yz\,(x\bar{w}+\bar{x}w)=0.$$

Starting with the eight denials in (1), Boole's conclusion may be readily obtained. What we want is an account of x *without* w. Now there is only one of the eight free from w as it stands[1], viz. the 8th; and it is easily seen that there are only two others which taken together can be freed from w, viz. the 1st and 3rd, for these combined yield $xyz=0$. Hence what we want is given by the two following:

$$\left.\begin{array}{l} \bar{x}\bar{y}z=0 \\ xyz\ =0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Add these together and we have

$$x\,(\bar{y}z-yz)=\bar{y}z$$

or

$$x=\frac{\bar{y}z}{\bar{y}z-yz}.$$

[1] For further discussion of this point see the next chapter.

Develop the right side, and we obtain

$$x = \bar{y}z + \tfrac{0}{0}y\bar{z} + \tfrac{0}{0}\bar{y}\bar{z};$$

or, more briefly,

$$x = \bar{y}z + \tfrac{0}{0}\bar{z} \quad \text{......................(3)},$$

which is Boole's conclusion, viz. "Wherever x is found there will either y be absent and z present, or z will be absent; and conversely, where y is absent and z present, there will x be present". This may be called, by comparison, the predicative reading of the result. Another reading of it would be "The class x consists of the class which is z but not y, and possibly 'some' of that which is not z".

(13) As another instance of an abbreviation in working take the following:—

$$\left.\begin{aligned} xy &= a \\ yz &= c \end{aligned}\right\}.$$

It is desired to obtain xz in terms of a and c, y being eliminated. The result may be got almost at once as follows:—

$$\begin{aligned} xz &= xz\,(y + \bar{y}) \text{ (developing with respect to } y) \\ &= xyz + x\bar{y}z \\ &= xy \,.\, yz + x\bar{y} \,.\, \bar{y}z \\ &= ac + x\bar{y} \,.\, \bar{y}z \text{ ....(1)}. \end{aligned}$$

Now, though $x\bar{y}$ cannot be expressed in terms of xy or a, it can in terms of $1 - xy$ or $\bar{a}$. For $1 - xy = x\bar{y} + \bar{x}y + \bar{x}\bar{y}$, so that $x\bar{y}$ may be described as an uncertain, or $\tfrac{0}{0}$ part of $1 - xy$ or $\bar{a}$. That is, $x\bar{y} = \tfrac{0}{0}\bar{a}$. Similarly $\bar{y}z = \tfrac{0}{0}\bar{c}$, so that $x\bar{y} \,.\, \bar{y}z = \tfrac{0}{0}\bar{a}\bar{c}$.

$$\therefore \text{ Finally } xz = ac + \tfrac{0}{0}\bar{a}\bar{c} \text{ ....(2)},$$

which is the answer desired.

(14) Develop the expression,

$$w = \frac{1 - \bar{x}\bar{y}}{x\bar{y} + \bar{x}y + z(xy + \bar{x}\bar{y})}.$$

The main difficulty of developing this by Boole's rule is the extreme liability to error in substituting 1 and 0 respectively for x, y, z, and their contradictories. Regarding the numerator and denominator as whole classes, we may write down the result at once $\left(\text{on the analogy of } \frac{a}{c} = a + \frac{0}{0}\bar{a}\bar{c}\right)$,

$$1 - \bar{x}\bar{y} + \tfrac{0}{0}\bar{x}\bar{y}\,(1 - x\bar{y} - \bar{x}y - zxy - z\bar{x}\bar{y})$$
$$= 1 - \bar{x}\bar{y} + \tfrac{0}{0}\bar{x}\bar{y}\bar{z},$$

whilst the condition implied in the fractional form of statement here becomes

$$(1 - \bar{x}\bar{y})\,\{1 - x\bar{y} - \bar{x}y - z\,(xy + \bar{x}\bar{y})\} = 0,$$

or, after multiplying out,

$$xy\bar{z} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

(15) The following example is, I think, the most intricate of any given by Boole:—(*Laws of Thought*, p. 146.)

1. Wherever x and z are missing, u is found, with one (but not both) of y and w.

2. Wherever x and w are found whilst u is missing, y and z will both be present or both absent.

3. Wherever x is found with either or both of y and u there will z or w (but not both) be found; and conversely.

They may be written

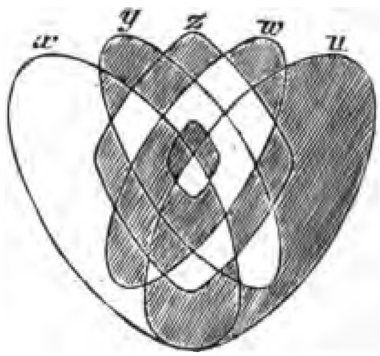
$$\left\{\begin{array}{l} \bar{x}\bar{z} = \tfrac{0}{0}u\,(y\bar{w} + \bar{y}w) \\ xw\bar{u} = \tfrac{0}{0}(yz + \bar{y}\bar{z}) \\ x(y + u) = (z\bar{w} + \bar{z}w) \end{array}\right. \quad \begin{array}{|l} yw,\ \bar{y}\bar{w},\ \bar{u} \\ y\bar{z},\ \bar{y}z \\ zw,\ \bar{z}\bar{w};\ \bar{x},\ \bar{y}\bar{u}. \end{array}$$

The appropriate factors being employed (they are written at the side) these equations resolve into

$$\left.\begin{array}{l} \quad\bar{x}y\bar{z}w,\ \bar{x}\bar{y}\bar{z}\bar{w},\ \bar{x}\bar{z}\bar{u}, \\ \quad xy\bar{z}w\bar{u},\ x\bar{y}zw\bar{u}, \\ \left\{\begin{array}{l} xyzw,\ xy\bar{z}\bar{w},\ xzwu,\ x\bar{z}\bar{w}u, \\ \bar{x}z\bar{w},\ \bar{x}\bar{z}w,\ \bar{y}z\bar{w}\bar{u},\ \bar{y}\bar{z}w\bar{u}, \end{array}\right. \end{array}\right\} = 0.$$

These thirteen denials contain all that the equations have to say, with some trivial redundancies. Some of them, as containing fewer terms, that is, as being less subdivided, are more comprehensive in their scope than others.

Now draw the 5-term diagram, and shade out the terms thus marked[1]. We have the following result:—



On looking at this diagram, several of the various conclusions which Boole has drawn are almost intuitively obvious. Thus that 'there is no xzw' ($xzw = 0$); that 'all w is either x or z' ($\bar{x}\bar{z}w = 0$); that 'all z is either x or w' ($\bar{x}z\bar{w} = 0$). These are the sort of conclusions to which diagrams specially lend themselves; for in each case we extinguish a connected group of classes, and each extinction readily catches the eye in a figure.

[1] We have not troubled to shade the outside of this diagram, viz. $\bar{x}\,\bar{y}\,\bar{z}\,\bar{w}\,\bar{u}$. (I may remark that, as Schröder and others have pointed out, the figure in the former edition was not quite accurate.)

Similarly it is not difficult to verify the conclusion that 'wherever x is found there will be found either z or w (but not both) or else y, z, and w will all be absent; and conversely' (Boole, p. 148). Symbolically this stands

$$x = z\bar{w} + \bar{z}w + \bar{y}\bar{z}\bar{w}.$$

On looking at the composition of x in the diagram it will readily be seen that it is made up of these three (in their ultimate subdivision, *six*) constituents. This sort of conclusion, though easy to verify by a figure, is probably easier to obtain from the symbolic letters. (It may be pointed out that the first of the three premises is redundant. We might omit the reference to w, and say simply 'wherever x and z are missing u and y are found'. Moreover, as Schröder remarks, there *is* no $xw\bar{u}$. The premises may therefore be expressed

$$\begin{cases} \bar{x}\bar{z} = \frac{0}{0}uy, \\ xw\bar{u} = 0, \\ x(y+u) = z\bar{w} + \bar{z}w. \end{cases}$$

These are exactly equivalent to Boole's premises.)

The reader will observe that this work of summing up the surviving classes into compendious propositions is that synthetic operation described at the end of the last chapter. If we adopt the plan there described we should begin by summing up the supplementary classes, viz. those not equated to zero, which are found to be

$$xyz\bar{w},\ x\bar{z}wu,\ x\bar{y}\bar{z}\bar{w}\bar{u},\ x\bar{y}z\bar{w}u,\ \bar{x}zw,\ \bar{x}y\bar{z}\bar{w}u.$$

These break up (in one mode) into the six factors

$$(x + z + \bar{w})(x + w + y\bar{z}u)(\bar{x} + z + w + \bar{u})$$
$$(\bar{x} + z + u + \bar{y}\bar{w})(\bar{x} + \bar{z} + y + \bar{w}u)(\bar{x} + \bar{z} + \bar{y} + \bar{w}) = 1.$$

Or, in words,

$\bar{x}\bar{z}$ is $\bar{w}$, $\bar{x}\bar{w}$ is $y\bar{z}u$, $x\bar{z}\bar{w}$ is $\bar{u}$, $x\bar{z}\bar{u}$ is $\bar{y}\bar{w}$, $x\bar{y}z$ is $\bar{w}u$, xyz is $\bar{w}$.

Slightly grouped they stand

$$\begin{cases} xz \text{ is } \bar{w}, \text{ and is } u \text{ or } y, \\ \bar{x}\bar{z} \text{ is } \bar{w}, \\ x\bar{z} \text{ is } wu \text{ or } \bar{w}\bar{u}\bar{y}, \\ \bar{x}\bar{w} \text{ is } y\bar{z}u. \end{cases}$$

Of course this is only one of a number of different arrangements which might be adopted. Judged by the mere number of letter terms involved, it is slightly briefer than Boole's own statement, though not so brief as his can be made, as we have shown above.

In discussing inverse forms we have naturally taken for consideration, in the first place, those in which the numerator of the fraction is definite, of which $\frac{x}{y}$ is the type. But in practice we may often find that they present themselves in what is really equivalent to the form $\frac{0}{0}x \div y$. The logical interpretation ought not to present the slightest difficulty. All that we were doing in the former case was to enquire for any class the part of which common to it and to y should be *identical with* x; whilst all that we are here doing is to enquire for any class the part of which common to it and to y shall be *included in* x, that is, shall 'be' x in the common predicative sense of the word. Or, stating the questions in the interrogative form, they stand thus;—

If z which is y is identical with x, find z. $\left(zy = x, \therefore z = \frac{x}{y}.\right)$

If z which is y is x, find z. $(zy = \frac{0}{0}x, \therefore z = \frac{0}{0}x \div y.)$

Whether the latter be stated in the form $zy = \frac{0}{0}x$, or in the form $zy = zyx$, it equally yields $zy\bar{x} = 0$,

$$\therefore\ z = \frac{0}{\bar{x}y} = \tfrac{0}{0}(1 - \bar{x}y),$$

$$= \tfrac{0}{0}(xy + x\bar{y} + \bar{x}\bar{y});$$

or, more briefly, $= \frac{0}{0}x + \frac{0}{0}\bar{x}\bar{y}$.

That is, in words, z may be 'anything that is x, or that is neither x nor y', under the assigned circumstances.

The reader should carefully observe that there is no condition implied here in the mere statement of the question, corresponding to the condition $x\bar{y} = 0$, in the case of $\frac{x}{y}$. That condition arose out of the demand that the restricted or common class should be *identical* with x; the mere *reference* of it to x makes no such requirement, provided of course we allow to x as a class term the customary freedom of range from *nothing* to *all* inclusive.

(16) Take a concrete instance. 'When all the books on philosophy are omitted from one of my bookcases there is nothing left but books on mathematics. What did the bookcase contain originally?' The answer is that it may have contained anything whatever excepting what was neither philosophy nor mathematics; an answer of course which is intuitively obvious.

Let z = contents of bookcase,

x = philosophy,

y = mathematics.

$$\therefore\ z(1 - x) = \tfrac{0}{0}y, \qquad z\bar{x}\bar{y} = 0,$$

$$z = \frac{0}{\bar{x}\bar{y}} = \tfrac{0}{0}(1 - \bar{x}\bar{y}).$$

(17) The continued repetition of the inverse operation will not give rise to any peculiar intricacies in Logic. No doubt such an expression as

$$\tfrac{0}{0}X \div x \div y \div z \div w$$

does not look the kind of thing of which Logic has been accustomed to take cognizance. Put it into words, however, and it is seen to be nothing but a very concise symbolic indication of "the class which on successive restriction by w, z, y, x, shall possess nothing but what is X". Let this class be Y, then we have

$$Yxyzw = \tfrac{0}{0}X,$$

$$Y\bar{X}xyzw = 0,$$

$$Y = \frac{0}{\bar{X}xyzw}$$

$$= \tfrac{0}{0}(1 - \bar{X}xyzw).$$

The answer therefore is that the only logical condition or determination of the desired class is that it must not contain "anything which is x, y, z, w, and which fails to be X". Whatever fulfils this condition will answer the purpose. Expressed as an ordinary disjunctive, we have

$$Y \text{ is } X \text{ or } \bar{x} \text{ or } \bar{y} \text{ or } \bar{z} \text{ or } \bar{w}.$$

I add one or two examples dealing with particular propositions.

(18) Show that a certain set of four properties must be found somewhere together, if the following facts are known :— "Everything that has the first property or is without the last has the two others; and if everything that has both the first and last has one or other but not both of the two others, then something that has the first two must be without the last two." (*Moral Sc. Tripos*, 1887.)

It will be simpler to throw the hypothetical premise into a disjunctive. ('If all a is β then some γ is δ' is equivalent to 'either some a is not β [i.e. there is $a\bar{\beta}$] or some γ is δ [i.e. there is $\gamma\delta$]'. And this latter is expressed by $a\bar{\beta}+\gamma\delta>0$.) The premises then become

$$\left.\begin{array}{l} x+\bar{w}=\tfrac{0}{0}\,yz\,;\ \text{i.e. } (x+\bar{w})(\bar{y}+\bar{z})=0 \\ xw\,(yz+\bar{y}\bar{z})+xy\bar{z}\bar{w}>0 \end{array}\right\}.$$

Now, assuming that the universals have the preference when there is a conflict, it is clear that, of the latter three admissible ultimate alternatives, the last two are excluded. Therefore the first must be admitted. That is, there must be $xyzw$, or the four properties must exist together.

(19) Given that 'either some x is y or some z is w', and also that 'either some x is z or some y is w'; find (1) the least amount of further assertion which will necessitate that x, y, z, w, shall all be represented; and (2) the maximum amount of further assertion which can be made without involving actual contradiction.

The first premise asserts that something must be saved out of xy, or something out of zw. Mark the list of combinations, thus respectively comprised, by the figures 1 and 2;

		z		$\bar{z}$	
		w	$\bar{w}$	w	$\bar{w}$
x	y	1,2 3,4	1,3	1,4	1
	$\bar{y}$	2,3	3		
$\bar{x}$	y	2,4		4	
	$\bar{y}$	2			

and those similarly comprised in the second pair of alternatives by 3 and 4.

The two premises combined contain four alternatives; and what they together assert is that some one compartment at least must be saved which is marked with 1 and 3, with 1 and 4, with 2 and 3, or with 2 and 4. There are five compartments so marked, and the saving of any one of these five will meet the requirements of the two premises. But of these five only one, viz. $xyzw$, will represent x, y, z, and w. Since then some one of these five must be saved, and only one of them will answer our purpose, the minimum amount of further assertion demanded will consist in denying the four which will not answer our purpose.

That is, if $xyz\bar{w}$, $x\bar{y}zw$, $xy\bar{z}w$, $\bar{x}yzw$ all $= 0$, then the two premises will necessitate that there shall be x, y, z, and w.

To express this demands the four ordinary propositions,

$$\begin{cases} \text{All } xyz \text{ is } w, \\ \text{All } xzw \text{ is } y, \\ \text{All } xyw \text{ is } z, \\ \text{All } yzw \text{ is } x, \end{cases}$$

which may be thrown into the single equational form

$$zw(x + y) = xy(z + w).$$

As regards the second part of the question, the only contradiction or inconsistency we recognize is that of attempting either to destroy all the combinations, or to save and destroy the same combination. Both of these are avoided so long as we spare the $xyzw$, or any one of the other four enumerated combinations. That is, the maximum amount of assertion that can be made consistently with the above premises may be expressed in one of the forms $xyzw = 1$,

$xyz\bar{w} = 1$, $xy\bar{z}w = 1$, $x\bar{y}zw = 1$, $\bar{x}yzw = 1$. Its symbolic expression, in either case, is short enough, but the volume or contents of its 'assertion', in our interpretation, is measured by the number of compartments which it clears out. This is obviously a maximum, comprising, as it does, 15 out of the total of 16.

CHAPTER XIV.

ELIMINATION.

Elimination is, both by its etymology and in its ordinary acceptation, the process of getting rid from within the limits of our enquiry of one or more of the symbols with which we were concerned at the outset. The reader will of course be more familiar with this term in mathematics than in Logic. That the process is resorted to in the common Logic, however, will be easy to show; and therefore in accordance with the general design of this work, we begin with it there. We shall thus be better able to trace the real nature of the process we propose to generalize, as the main characteristics of the Symbolic system rather tend to disguise the substantial identity between the rudimentary and the developed forms which elimination may assume.

Beginning then with immediate inferences, look at the step which might be called "inference by omitted determinants[1]". When this is interpreted in respect of the extension

[1] Inference by *added* determinants is already recognized (e.g. Thomson; *Laws of Thought*, p. 158). But the distinction introduced by *omitting* a portion of the predicate might just as well be considered to constitute a new judgment as that introduced by adding on a new portion. Bain (*Deductive Logic*, p. 109) denies that there is any inference here.

or denotation of the terms involved, it is an exact case in point, of a simple kind, of the process of which we have to give an account. 'Greeks are rational mortals; therefore they are mortals':—Here we have omitted the term 'rational' from our result, that is, we have eliminated it. Or we might have omitted the word 'mortal' by saying that 'Greeks are rational'.

So again the syllogism is a case of the elimination of a middle term. Viewed in its extension, as an arrangement of classes, the mood *Barbara* asserts that a class contained within a second class is contained within the wider class which contains this second one; the reference to this second, or middle term, being omitted from our result: that is, it is eliminated. The *Dictum* of Aristotle may in fact be regarded as a formula of elimination for the simple groups of propositions to which it applies. This is obvious enough in the case of the affirmative form of the Dictum, as in Barbara, but may be seen with almost equal ease to be so also in the negative form. For instance, 'No Y is Z; all X is Y; therefore no X is Z'. Here we say in effect that Y is included in not-Z, i.e. is a part of it; and X is included in Y; therefore X is included *à fortiori* in not-Z. On the view of terms and propositions which is adopted in our system, and which is a rigid class-view, this negative form of the Dictum is therefore the precise equivalent or formal reproduction of the affirmative form. Not-X and not-Z are, with us, classes of exactly the same kind and significance as those which we designate by X or Z, and there is no difference of principle in referring sub-classes to one or other of them. In both cases the middle term, or class which at once includes and is included, is omitted or eliminated.

Now the characteristic of this Elimination to which I wish prominently to direct the reader's attention, as con-

taining the main clue to its significance in Logic, is this:—that we have substituted a *broader or less exact determination* in the place of the one which was first given to us. That is, we have had to let slip a part of the information contained in the data.

That this is so in the case of the immediate inference is abundantly clear; for the Greeks who were before referred to the class of 'rational mortals' are now referred to an uncertain part of the larger classes 'rational', or 'mortal'. The reason why it is not equally clear in the case of the syllogism is that the premises are given to us separately instead of being combined into one. That is, in the former case we say 'z is xy, therefore (more vaguely) z is y or x'; whilst in the latter we say 'z is x, and x is y, therefore z is y'. Of course the distinction between mediate and immediate inference is on various grounds important, both of speculation and of logical procedure; and nothing here said is meant to gloss over that distinction in its due place. All that is now asserted is that in each case alike, whether there be one premise or two, the full implied reference of z was to xy, and that consequently the statement that it is simply y (in other words, the elimination of x) is so far a less exact determination of it than could be given by the retention of that term.

This loss of precision in the process of elimination is the general result, but a certain narrow class of exceptions can be pointed out. When x and y are coextensive the substitution of one for the other leaves the extension unaltered. Thus in the elimination of y from 'all x is all y, all y is all z' the determination of x as z is just as narrow as that given by calling it yz. But of course such a case as this can but rarely occur. It is not generally easy, except when we are dealing with definitions, to find two terms thus co-

extensive; and the occurrence of three such must be very rare indeed.

It will be understood that this loss of precision is no valid objection to the process of elimination. It is one of the many characteristic distinctions of the class-explanation of propositions that we thus clearly call attention to the fact that there is any such loss at all. On the common explanation we only think of the major term in its capacity of a predicate, and we want to know whether or not it is to be attached to the subject. The middle term is used presumably merely as a means toward deciding this fact, and when it has answered its purpose it is very properly dropped from notice. We only wanted to prove, say, that z is y; to insist upon it that z is xy, though quite true, may be needless trouble. The very words *Discourse* and *Discursive reasoning* seem to point to this. We let the mind run from one thing to another; and we only dwell finally upon, and put into our conclusion, the particular fact or facts which we happen to need. We distinctly want to get rid of the middle term, and not to carry all our knowledge about in our predicate. It is of the essence, on the other hand, of the Symbolic system, to keep prominently before us every one of the classes represented by all our terms and their contradictories. Accordingly the distinction between xy and y, and the fact that the latter must generally speaking be a broader and looser determination, is much more prominently set before us here.

The fact is that here, as in various other directions, the associations derived from the mathematical employment of the term are apt to be somewhat misleading. For one thing we are accustomed to believe that there must be some connexion between the number of equations set before us, and the number of terms involved in them, for the purpose of

elimination; so that one term demands for its elimination two equations, and so forth. In Logic on the other hand we know that the number of statements into which we throw our data is largely a matter of our own choice, a single logical equation admitting of equally ready statement in the form of a group of several. Accordingly the number of equations at our command in no way affects the question of the possibility of elimination here. Again, as regards the loss of precision: in mathematics the case is rather the other way. If we have three equations connecting x, y, z, each of these may be conceived to represent a surface, which is satisfied by a doubly indefinite number of values of these variables. But if I eliminate y and z, I obtain one or more determinate values of x corresponding to the particular points where the surfaces intersect. We have gained, in the process of elimination, an increase of definiteness which must be estimated as one of kind rather than of degree.

What we have now to do is to see how the logical process which has been illustrated in one or two simple instances can be generalized. As an easy example begin with the following,—

$$w = xy + \bar{x}z$$

and suppose we are asked to eliminate y from it. As an equation its significance is plain enough. It is nothing else than a definition or description of w in terms of x, y, and z. Any one therefore who knows the meaning of these terms, or the limits of the classes for which they stand, will have all the information which they can furnish him as to the meaning or limits of w. Assuming that we are confined to the use of the three terms x, y, z, then w is as precisely determined as circumstances permit.

This being so, what could be meant by 'eliminating' y

from the equation? If we are not to retain it there, and are not to introduce some new equivalent for it, the only remaining course is to do as well as we can without it. But it cannot be simply omitted; for this would be inaccurate, unless we took care to indicate somehow that we had dispensed with it. Apparently, therefore, all that is left for us to do is to take refuge in the vague, and to substitute for y, wherever it occurs, some such word as 'some'. If we did this we should just write the equation in the form

$$w = \text{'some'}\, x + \bar{x}z$$

$$\text{or } w = \tfrac{0}{0}\, x + \bar{x}z \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

This method of elimination has at least the merit of frankness. It points out where we have let go some of the determining elements, and it indicates exactly the nature and amount of the consequent introduction of vagueness. There is another equivalent form to this, which consists in writing the altered equation thus:

$$w = wx + \bar{x}z.$$

This disguises the real vagueness under a show of information; and offers us an implicit equation involving w, for the explicit description of w with which we started.

Nothing could better show the nature of logical elimination than this simple example. The term y was one of the elements employed in the determination of w; hence its abandonment will necessarily entail some loss of precision. If we were dealing with real equations of the mathematical type such loss would generally be fatal to the value of our conclusion. But what we have to do with in Logic is rather the subdivision of classes by other class terms, and the identification of a group of individuals under different class designations. Hence the letting slip of one of our class terms

will only refer us to a somewhat vaguer and wider class than that with which we started. The relinquishment of y does not destroy all the knowledge of w with which we started, but it certainly destroys a part of it.

If the only logical statements with which we were concerned were of this simple type—in which we have a term standing by itself on one side, with a description or definition of it on the other,—no other plan of elimination would be necessary. But, as the reader knows, we have to encounter much more complicated statements than this, viz. those in which every term is implicitly involved. So we must look out for some more general mode of elimination. We should best seek for it in the alternative *negative* interpretation of our propositions. We know that every logical equation can be thrown, without any loss whatever in its significance, into the shape of a number of distinct and peremptory denials.

Take then, to begin with, the same statement as before and look at it, for comparison, in the light of what it denies. Adopting the plans described in former chapters, we find that it may be thrown into the form of the five following denials;—

$$\left.\begin{matrix} wx\bar{y} \\ w\bar{x}\bar{z} \\ w\bar{y}\bar{z} \end{matrix}\right\} = 0. \qquad \left.\begin{matrix} \bar{w}xy \\ \bar{w}\bar{x}z \end{matrix}\right\} = 0.$$

Now setting before us the same aim as in the preceding example, viz. of determining w as well as we can without making use of y, the course we should naturally adopt would be this;—we should just omit those amongst the above denials which involve y or $\bar{y}$, retaining the remainder which do not. If we did so the result of the elimination would stand thus;

$$\left.\begin{matrix} w\bar{x}\bar{z} = 0 \\ \bar{w}\bar{x}z = 0 \end{matrix}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

That this form is the precise equivalent of (1) is easily seen. For multiplying both sides of (1) by $\bar{x}\bar{z}$ and $\bar{w}$, it resolves into these denials and into no other unconditional ones. Similarly on combining and developing (2) we have $w = \bar{x}z + \frac{0}{0}x$ as in (1).

These two simple methods of elimination are therefore precisely equivalent, at any rate in this instance. The only difference between them is, that, whereas the original determination of w was complete and accurate, (1) retains as far as possible the same form, marking plainly the position and limits of the lacunæ of information caused by the omission of y, whilst (2) contents itself by giving the materials for this latter statement.

Take again such a case as the following:—

$$w = xy\bar{z} + x\bar{y}z + \bar{x}yz$$

in which w is declared to comprise 'those things which possess two, and two only, of the attributes denoted respectively by x, y, z'. Let it be required to eliminate y, a term (be it observed) which enters into every element of w.

Form (1) would express the result off-hand[1] as

$$w = \tfrac{0}{0}x\bar{z} + \tfrac{0}{0}xz + \tfrac{0}{0}\bar{x}z$$

$$\text{or } w = \tfrac{0}{0}x + \tfrac{0}{0}z \text{ ....(1)}$$

a result in which we have had to depart some way from the original determination. All that we can substitute for that determination without appeal to y, being, that 'w is contained somewhere within the limits of x and z':—that is, there is certainly no w outside that boundary.

[1] In accordance with what has been pointed out more than once we must write $\frac{0}{0}$ for $\bar{y}$ as well as for y. For the limits of indefiniteness of any term and its contradictory are the same, so that $1-\frac{0}{0}$ is the same as $\frac{0}{0}$ when used as a symbol of indefiniteness. Each is entirely uncertain between 0 and 1.

The other form would have looked to the denials involved in the equation, and the selection of those amongst them which do not make use of y. There is a very simple way of effecting this in practice. Instead of using all the factors which will disintegrate the given equation, and then selecting only the elements we want, we had better only make use of those factors which we see will produce these latter. Thus, here, $\bar{x}$ and z would make $xy\bar{z}$ vanish but not the other two terms; $\bar{x}$ and $\bar{z}$ will make the second vanish, and x and $\bar{z}$ the third. The only factor therefore that will make all the three y-terms disappear will be $\bar{x}\bar{z}$. Hence the only elementary denial which can be found without involving y will be

$$w\bar{x}\bar{z} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and this is the required elimination. As in the former case it may readily be shown that these two results are precisely equivalent and deductively interchangeable.

Before pointing out the practical or theoretical defects of these methods it will be worth while to apply them to a group of statements. As already insisted on, there is no distinction of principle between the information conveyed by one, and by a plurality of statements; but the striking difference in this respect between the logical and the mathematical calculus deserves emphatic notice.

Take the following:—

$$\begin{cases} \text{Every } w \text{ is either } x \text{ and } y, \text{ or } z \text{ and not } x. \\ \text{Every } w \text{ is either } x, y, \text{ and } z, \text{ or neither } x \text{ nor } z. \end{cases}$$

In symbols they stand thus

$$w = \tfrac{0}{0}(xy + \bar{x}z)$$
$$w = \tfrac{0}{0}(xyz + \bar{x}\bar{z})$$

Let it be required to eliminate y from these two statements.

The simplest plan would be to commence by multiplying the two together, when we get $w = \frac{0}{0}xyz$. Substitute $\frac{0}{0}$ for y, and remember that $\frac{0}{0} \times \frac{0}{0} = \frac{0}{0}$, (for no such multiplication can alter the range of indefiniteness of the symbol) and we have w with the desired elimination

$$w = \frac{0}{0}xz.$$

If we had begun by eliminating separately from each, we should have had

$$\begin{cases} w = \frac{0}{0}(x + \bar{x}z) \\ w = \frac{0}{0}(xz + \bar{x}\,\bar{z}) \end{cases}$$

the combination of which would lead to the same result as above.

Or had we broken them up into their respective denials, and added these together, we should have been led to the following (omitting those which involve y)

$$w\bar{x}\bar{z} + w\bar{x}z + wx\bar{z} = 0$$

which leads again to

$$w = \frac{0}{0}xz.$$

If all examples resembled the simple ones discussed above we should need no other methods of elimination than those just described. But the former method is only properly available when we are dealing with equations of an *explicit* kind; so that if our statements were not originally in that shape we should have to reduce them to it. As regards the second there is a cause of possible failure, unless we are on our guard, which will deserve notice.

The difficulty arises as follows. We have given directions to break up the equation into its ultimate denials, and then to select those amongst them which do not involve the term to be eliminated. And, in the simple examples which we took, such terms presented themselves at once. But it is

easily seen that none such may be found; in fact, if we have developed every element to the utmost extent, none such *can* be found, for every term will then have been subdivided into its y and not-y parts. Thus in the example on p. 366 we found the two denials $w\bar{x}\bar{z}=0$, $\bar{w}\bar{x}z=0$, which did not involve y, and we chose them accordingly. But if these had presented themselves, as they might, in the forms $w\bar{x}\bar{z}y=0$, $w\bar{x}\bar{z}\bar{y}=0$, $\bar{w}\bar{x}yz=0$, $\bar{w}\bar{x}\bar{y}z=0$, there would have been apparently no terms free from y.

We have therefore to amend our rule. We must say that the complete results of the elimination of any term from a given equation are obtained by breaking it up into a series of independent denials, and then selecting from amongst these all which either do not involve the term in question, or which *by grouping together can be made not to involve it.*

The foregoing results may be summarized in the following formula. Starting, as above, with the sum-total of denials involved in the premises, group these according as they involve x, $\bar{x}$, or neither. The result will stand,

$$Ax + B\bar{x} + C = 0.$$

Now, by supposition, C being free from x and $\bar{x}$ is an instalment towards the desired elimination result. The remainder will be included by the common part of A and B; for since this involves both x and $\bar{x}$ the addition of these two constituents will give a result free from either x or $\bar{x}$. Therefore $AB+C$ contains all the terms free from x, or which can be freed from it. But $AB=0$ (for No A is x, and All B is x) and C clearly $=0$. Therefore the desired result, viz. the remains of the original equation after x has been eliminated, is $AB+C=0$. This is the process of elimination as described by Professor Schröder, and sometimes called his modification of Boole's method.

The process of elimination in the case of particular propositions is conducted upon the same general principles, but differs somewhat in result. We may take, as the typical corresponding type,

$$Ax + B\bar{x} + C > 0.$$

What this asserts is that something within the range of Ax or $B\bar{x}$ or C has to be saved. Now it is obvious that whatever lies within the range of Ax lies within that of A:— the inference here is nothing more than that which is familiar in Logic under the name of 'inference by omitted determinants', that is, by simple abstraction. We may therefore simply substitute for the above inequation,

$$A + B + C > 0,$$

which is the elimination required.

If the particular propositions in question are to be combined with universals, then we may express the above elimination in a somewhat narrower and therefore more determinate manner. The universals, if one may so put it, have their say first, and thereby destroy a number of possible combinations. Taking account of these we may introduce a measure of determination into A, B, C.

Suppose, for instance, we have, in addition to the above,

$$Dx + E\bar{x} + F = 0.$$

We know that the elimination of x from this yields $DE + F = 0$. Also that 'All $\bar{x}$ is $\bar{E}$' and 'All x is $\bar{D}$'; D, E, and F, being of course functions of y, z, &c., and therefore entirely independent of x. Instead therefore of simply omitting x and $\bar{x}$ in the inequation, we may substitute their value as given in the equation, and write

$$A\bar{D} + B\bar{E} + C > 0,$$

which is of course a narrower and more determinate elimination result than we obtained before.

Generally speaking, if the result of a system of equations is given in the form

$$\left.\begin{array}{l} Ax + B\bar{x} + C = 0 \\ Dx + E\bar{x} + F > 0 \end{array}\right\},$$

the complete elimination of x is given by

$$\left.\begin{array}{l} AB + C = 0 \\ D\bar{A} + E\bar{B} + F > 0 \end{array}\right\}.$$

There are one or two characteristic points of difference between the two kinds of elimination which deserve notice. We have seen that elimination is not always possible in the case of universals; in fact that a large total amount of assertion may be made without its necessarily following that such a process is possible. If, for instance, we have $Ax + B\bar{x} = 0$, we know that provided A and B keep clear of each other,—i.e. up to the limit of their together making up the universe,—there can be no elimination of x. But from the corresponding expression $Ax + Bx > 0$ we have at once $A + B > 0$, whatever A and B may be. That is, from every such group of terms, x may be always eliminated.

This points to the case, in elimination of particulars, in which we obtain a result which is quite true but illusory[1]. For if A and B together make up the universe the expression

[1] A corresponding result presents itself on Dr Mitchell's plan of notation. As I have pointed out, his standard expression for the universal proposition is $f(x) = 1$, say,

$$Ax + B\bar{x} + C = 1:$$

i.e. he starts with equation to unity, or the universe. Here x may be eliminated, just as we have done above, in the case of particulars, by its mere omission. Thus, in this example, the elimination of x yields

$$A + B + C = 1.$$

If any one or more of these terms, or their sum, $=1$, the result is illusory.

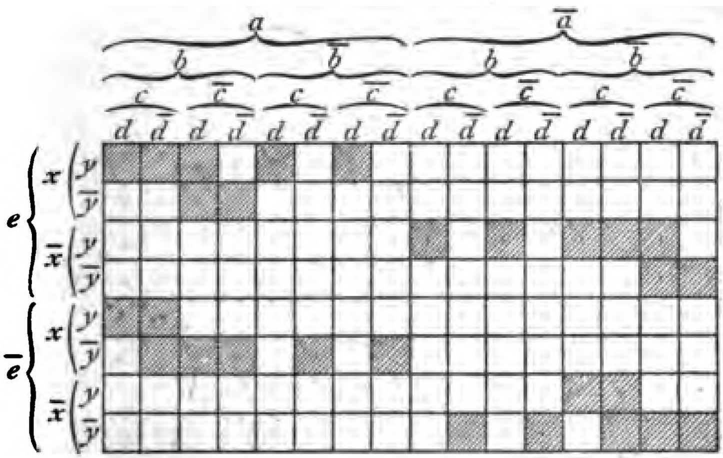
$A+B>0$ tells us nothing; for, as often remarked, the postulate that something in the universe exists is our one condition of consistency. The conditions therefore for possibility of effective elimination in the case of universals and particulars tend in opposite directions. If A and B are not extensive enough to encroach upon each other's ground there can be none of the former; if they are so extensive that between them they cover the universe there can be none of the latter. The critical intermediate case is when A and B are contradictory opposites ($A=\bar{B}$): in that case neither kind is possible. Thus the expression,

$$(xy+\bar{x}\bar{y})\,z+(x\bar{y}+\bar{x}y)\,\bar{z},$$

whether $=0$ or >0, will yield no elimination of z to any effective purpose.

The nature of this process of elimination may be conveniently illustrated by one of the diagrams described on page 140. Take, for instance, the denials[1]

$$\left.\begin{array}{l}(abx+\bar{a}\bar{b}\bar{x})\,(cy+\bar{c}\bar{y})=0\\(dey+\bar{d}\bar{e}\bar{y})\,(ax+\bar{a}\bar{x})=0\end{array}\right\}.$$



[1] The problem from which these are derived will be discussed in a future chapter.

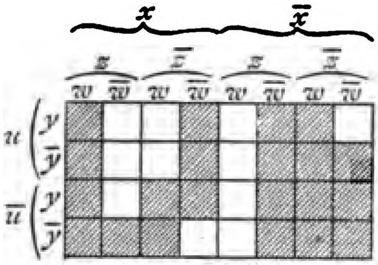
Erase the compartments included in these denials, and we have the above result. Now the elimination of any term, y, is equivalent to finding among the destroyed classes two which differ only in containing y and $\bar{y}$; for these may be run into one (since $y+\bar{y}=1$), and thus described without employment of y or $\bar{y}$. It is seen at once that this is equivalent to finding two classes vertically adjacent, taken one each from the 1st and 2nd, 3rd and 4th, 5th and 6th, or 7th and 8th horizontal lines. Clearly there are only four such, viz. $ab\bar{c}dex$, $abc\bar{d}\bar{e}x$, $\bar{a}\bar{b}\bar{c}de\bar{x}$, $\bar{a}\bar{b}c\bar{d}\bar{e}\bar{x}$. The equation of these to zero *is* the elimination of y. To eliminate x we have to perform a similar process, making our selection of the pairs from the 1st and 3rd, 2nd and 4th, 5th and 7th, 6th and 8th, horizontal lines respectively. It is plain that no such selection can be made. Therefore x cannot be eliminated. To eliminate e, again, we must take squares vertically over each other but *four* horizontal lines apart. Of these there are eight altogether. To eliminate a, b, c, or d, we adopt the same plan, merely substituting 'horizontal' for 'vertical' and *vice versâ*. It is easily seen that a cannot be thus got rid of, but that b, c and d can.

It will, of course, save trouble in these cases to arrange the terms to be eliminated (when we are dealing with more than one of them) on the same side of the figure. Thus, take the following example. (It is that of Boole; already discussed on p. 351.)

$$\begin{cases} \bar{x}\bar{z}(\bar{u}+wy+\bar{w}\bar{y}) & =0 \\ \bar{u}xw(y\bar{z}+\bar{y}z) & =0 \\ x(u+y)(zw+\bar{z}\bar{w}) & =0 \\ (\bar{x}+\bar{u}\bar{y})(z\bar{w}+\bar{z}w) & =0. \end{cases}$$

Let it be required to eliminate u and y, and then to determine x.

Shade out, according to the premises, and the question of the elimination of u and y then becomes simply the



detection of those vertical columns which are entirely obliterated. That is, the desired solution is

$$xzw + \bar{x}z\bar{w} + \bar{x}\bar{z}w = 0.$$

With this information we have

$$x = z\bar{w} + \bar{z}w + \tfrac{0}{0}\bar{z}\bar{w}.$$

Take another instance, in which particular propositions are involved[1].

Given $$\left.\begin{array}{l} x = \bar{c}y + e\bar{y} \\ z \neq c\bar{w} + \bar{e}w \end{array}\right\}.$$

Find the relation between x, y, z, w, after elimination of c and e[1].

Remembering that the equation $x = y$ is resolvable into the collective destructions $x\bar{y} + \bar{x}y = 0$, and the inequation $x \neq y$ is resolvable into the alternative conservations

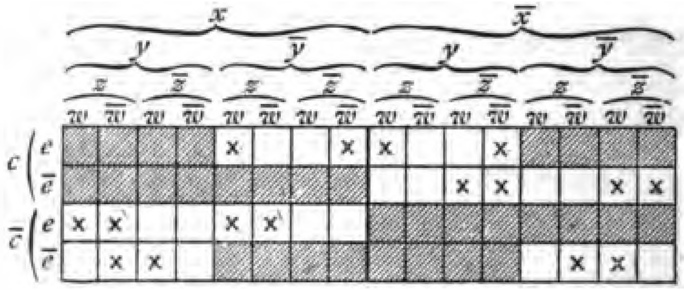
$$x\bar{y} + \bar{x}y > 0\,;$$

the above data may be written down

$$\left.\begin{array}{l} x(cy + \bar{e}\bar{y}) + \bar{x}(\bar{c}y + e\bar{y}) = 0 \\ z(\bar{c}\bar{w} + ew) + \bar{z}(c\bar{w} + \bar{e}w) > 0 \end{array}\right\}.$$

[1] Proposed by Mrs Ladd-Franklin. See Schröder, II. 309, where a solution, by another method, is given: I agree with his result, which differs from that of the others who offered solutions.

Erase the compartments all of which are to be destroyed,



and mark with a cross those of which some one or more are to be saved.

Beginning with the negations, it is clear that these can furnish no elimination such as we want; since for that purpose a whole vertical column would have to be erased. In no other way can c and e both be dispensed with. Accordingly no *universal* proposition can thus be inferred.

Turn next to the conservations. Of these there are 16. There would have been 32, but for the preliminary action of the universal premises in blotting out half of the total number involved in the particular premises. Now just as in the last case what we had to do was to select all those vertical columns (if any) which we saw to be wholly destroyed, so here what we want to do is to select all those out of which some one or more elements are to be saved. Of these there are 12, and their enumeration is the elimination desired. They may be summarily described by the expression

$$xz + \bar{x}\bar{z} + yw + \bar{y}\bar{w} > 0;$$

for these terms comprise all the 12 columns in question. (This result is obtained by either enumerating them all in detail, beginning with $xyzw$; or, indirectly, by e... rating the remaining four, and 'contradicting' the s... these.)

Two points deserve notice here. In the first place we see what exactly the universal premises have done. They are not ineffective, as might be hastily supposed. But for them *every* column would have had to be enumerated, and we should have obtained only the unmeaning result that some one or more of the 16 possible combinations of x, y, z, w, was to be saved: a result which is *à priori* necessary. In the second place it must be noticed that elimination, where particulars are concerned, is less completely stated than where universals are concerned. It will be seen that in four of the columns *all* the surviving squares are marked, in the remaining twelve some only are marked. This may be stated in words by saying that in the case of these four columns ($xyz\bar{w}$, $x\bar{y}zw$, $\bar{x}y\bar{z}\bar{w}$, $\bar{x}\bar{y}\bar{z}w$) we must be prepared to meet the 'survival' in any one of the compartments left undestroyed by the first premise, but that in the case of the other 12 columns the second premise gives us no right to expect it in any but a portion of those compartments. Had such a distinction occurred in the case of ordinary elimination we should have marked it by our notation. If, for instance, we had been asked to determine x after elimination of e, we should have expressed it

$$x = \bar{c}y + \tfrac{0}{0}\bar{y}.$$

This is a closely parallel case to the above, and there seems no reason why we should not indicate each in the same manner.

As an illustration take the following example:—Either some a that is x is not y, or all d is both x and y. Either some y is both b and x, or all x is either not-y or both c and not-b. Find the solution, after elimination of x and y. (Proposed by Mr Mitchell, in the Johns Hopkins *Studies*, p. 85.* In *Mind*, 1884, I offered a solution which practically

agrees in result with the following, but would require too much space for its justification here.)

The premises, with the consequent eliminations, may be arranged as follows:—

Premises.	*Eliminations.*
$a x \bar{y} > 0 \begin{cases} bxy > 0 \\ xy(b+\bar{c}) = 0. \end{cases}$	$a > 0$ and $b > 0$ $a > 0.$
$d(\bar{x}+\bar{y}) = 0 \begin{cases} bxy > 0 \\ xy(b+\bar{c}) = 0. \end{cases}$	$b > 0$ $d(b+\bar{c}) = 0.$

The significance of this arrangement will be readily understood. The first of the four resultant alternatives on the left side is that of two particular premises, $ax\bar{y}>0$, $bxy>0$; in which the elimination of x and y yields $a>0$ and $b>0$. The second is that of a particular and a universal; in which the elimination of x and y from the former element gives only $a>0$. But clearly x and y cannot be eliminated from the latter, or universal, therefore we have no destructive result, and no correction to introduce into the conservation. It must therefore stand $a>0$. The third again yields an exactly similar result, with the only difference of $b>0$ instead of $a>0$. The fourth alternative is the familiar case of the combination of two universals; and yields the elimination,

$$d(b+\bar{c}) = 0.$$

These four eliminations are themselves of course disjunctive alternatives. Their results may be verbally expressed by the statement, Either there must be a and b, or a, or b, or no $d(b+\bar{c})$. The first of these cases being obviously included in the second and third, we may say

simply, Either there must be a or b, or no $d(b+\bar{c})$. In symbols,

$$\text{Either } a \text{ or } b > 0, \text{ or } d(b+\bar{c}) = 0.$$

When so stated there is a slight redundancy of expression; for, in the case of there being no a or b, it is sufficient to say simply $d\bar{c} = 0$. That is, Either a or $b > 0$, or $d\bar{c} = 0$.

CHAPTER XV.

THE EXPRESSIONS $f(1)$ *AND* $f(0)$[1].

THE expressions $f(1)$ and $f(0)$ are presumably, to the bulk of logicians, the most deterrent of the various mathematical adaptations of which Boole made use in his system. They play too prominent a part however in that system to admit of neglect; and indeed on their own account they deserve study, as the effort to detect the logical significance of such abstract symbolic generalizations as these seem to me one of the most useful mental exercises which the study of the subject can afford. And that these peculiar expressions are really nothing more than generalizations of very simple logical processes will soon be made manifest.

We will examine these expressions under two heads; firstly as mere class symbols, and secondly as resulting in logical equations. That is, we will begin by taking them as derivatives of $f(x)$, and secondly as derivatives of $f(x) = 0$.

[1] The merely logical reader will not find the study of this chapter essential to the comprehension of those which follow.

Take then the expression

$$xz + \bar{x}y + w\bar{y}\bar{z}$$

and calling this $f(x)$, examine the significance of $f(1)$ and $f(0)$.

Symbolically, of course, the answer is prompt. Write 1 for x all through, and we have $f(1) = z + w\bar{y}\bar{z}$; write 0 for x all through, and we have $f(0) = y + w\bar{y}\bar{z}$. But what we want is the logical interpretation. To obtain this it is only necessary to remember that xz means 'z restricted by x', and $\bar{x}y$ means 'y restricted by not-x'. But 1, as a factor, means "the whole of", so that the substitution of 1 for x is merely the direction to take the *whole* of z instead of only the x-part of it[1]. Similarly 0 meaning "none of", the substitution of 1 for x, in $\bar{x}y$, tells us to take *no* y instead of the not-x part of it. Hence the exchange of $f(1)$ for $f(x)$ is only the generalized symbolic direction:—Go through the given class expression; and of every element of it which is limited by x take the whole, and of every element limited by not-x take none, and let the terms which do not involve x or $\bar{x}$ remain unaltered.

This indicates the equally simple logical explanation of $f(0)$. The latter expression is in a way the exact converse of $f(1)$; that is, whatever we there did with x we here do with not-x, and *vice versâ*. Hence what we are here directed to do is to secure that every term which involves x shall just be dropped out, that every term which involves not-x shall be taken in its full extent instead of under this restriction,

[1] The reader will remember the distinction between this mere *taking off* the restriction of x, by turning xz into z, and the true inverse operation to its imposition. Restrict x by z and we have simply xz: take off this x-restriction from xz and we have x: but the inverse to xz, *i.e.* the class which will become xz when the condition of x is imposed upon it, is of course $xz + \frac{0}{0}\bar{x}z$, in its fullest terms.

and that every term independent of x shall be simply let alone.

In the above explanation we have implied that there may be terms in our expression which do not involve either x or not-x. But this need not be so, and if the expression were fully developed it could not be so, for every term would then be divided respectively into its x and its not-x part. The term $w\bar{y}\bar{z}$ is so put for brevity merely, it is really equivalent to $w\bar{y}\bar{z}(x+\bar{x})$. When therefore any logical class expression is completely broken up into its ultimate elements all these will fall into two ranks, those of x and not-x. The verbal statements of $f(1)$ and $f(0)$ then become simplified. The former says, Take the x-members unconditionally, and discard those which are not-x: the latter says, Take the $\bar{x}$-members unconditionally, and discard those which are x.

Next as regards the relation of these expressions, $f(1)$ and $f(0)$, to each other. It might be hastily inferred that they are complementary to one another, so that they should be mutually exclusive and should together make up $f(x)$;—that is, that

$$f(1)+f(0)=f(x), \text{ and } f(1)f(0)=0.$$

Closer observation however will show that neither of these results need hold good, and that the former generally will not: but it will be well to discuss these relations in detail, as they have an important bearing upon the problem of Elimination.

Suppose then that $f(x)$ takes the form $Ax+B\bar{x}$, where A and B are combinations involving y, z, and the other class terms which enter into the given expression. We know that $f(x)$ must be representable thus, for when expanded fully it can only yield x and not-x terms, and the factors of these terms can only be composed of various combinations of the

other class terms. Then $f(1)=A$, and $f(0)=B$; that is, $f(1)$ is a wider class, restricted by x; and $f(0)$ a wider class, restricted by not-x.

What then are the limits as to class extension of A and B? None necessarily, except that they must conform to the fundamental law of logical classes, viz. that neither of them can exceed unity. In the extreme limiting case of A and B being both equal to 1, we should then have taken the whole of x and the whole of not-x, so that our $f(x)$ would have itself to equal 1. That is, when $f(1)$ and $f(0)$ each equal 1, then $f(x)=1$. (Thus let

$$f(x)=xy+x\bar{y}+\bar{x}y+\bar{x}\bar{y}=1;$$

then $f(1)=y+\bar{y}=1$, $f(0)=y+\bar{y}=1$.) An intermediate case is when $A+B=1$, that is, when the aggregate of the factors of x and $\bar{x}$ together make up the universe. In this case, if A and B are exclusives, $f(1)$ and $f(0)$ are contradictory opposites. (Thus let

$$f(x)=x(y\bar{z}+\bar{y}z)+\bar{x}(yz+\bar{y}\bar{z});$$

then $f(1)=y\bar{z}+\bar{y}z$ and $f(0)=yz+\bar{y}\bar{z}$, so that

$$f(1)+f(0)=1.)$$

We gather then that $f(1)$ and $f(0)$ may, as regards the extent of ground they occupy, just make up between them the whole universe. But they may also, in one direction, both of them shrink to zero (in which case $f(x)=0$); or, in the other direction, extend till both are unity (when $f(x)=1$).

Similarly as regards the product of these expressions, viz. $f(1)\,f(0)$. Its value depends upon the mutual relation of these aggregate class factors of x and $\bar{x}$ in the subdivision of $f(x)$. If A and B are entirely composed of mutually exclusive elements, whether or not these make up the universe

between them,—that is, if each of those in A is exclusive of each of those in B,—then AB, or $f(1)\ f(0)$, must $= 0$; i.e. $f(1)$ and $f(0)$ are classes which are mutually exclusive of each other.

We may sum up therefore by saying that, given $f(x)$ as a true logical expression for a class group, then $f(1)$ and $f(0)$ will also represent class groups. As regards their mutual relations to each other and to the original $f(x)$, we may lay down the following conclusions (omitting various limiting cases which the reader will readily work out with the help of examples):—Each of these expressions, $f(1)$ and $f(0)$, will omit a portion of what was included in $f(x)$, but will also contain a portion of what was not included in it. And as the portions of $f(x)$ which they thus omit will not be the same portion (one being an x and the other a not-x portion) it is certain that between them they must at least cover the whole of $f(x)$, besides including something else. That is, $f(1)$ consists of the whole of A, and $f(0)$ of the whole of B. But $f(x)$, or $Ax + B\bar{x}$, comprises only a part (the x-part) of A, and only a part (the not-x-part) of B; so that $f(1)$ and $f(0)$ must between them cover all $f(x)$, and may cover any part of it twice, besides covering once or twice any part of what is not $f(x)$. On the other hand it is possible that $f(1)$ and $f(0)$ may be classes entirely exclusive of each other.

Now for the far more important case of logical equations. Nearly all that has been said above will hold good in this case, for every logical equation can be expressed in the form of a class group of a peculiar kind. That is, it may be interpreted as declaring that a certain aggregate of classes is, collectively and individually, non-existent. Take for instance the statement $w = xy + \bar{x}z$. Develop and rearrange it, and it stands thus,

$$wx\bar{y} + w\bar{x}\bar{z} + \bar{w}xy + \bar{w}\bar{x}z = 0,$$

this being in every essential respect, as regards assertion and denial, the exact equivalent of $w = xy + \bar{x}z$. The two forms are really the same statement differently arranged. Hence it is clear that $f(x) = 0$ is merely a particular application of $f(x)$; and everything that has been said above about the interpretation of $f(1)$ and $f(0)$ and their mutual relations to one another will hold good here also. The form, and the detailed meaning, of a class expression, are entirely unaffected by our having to add that no such class is in existence, which is the only difference introduced by the equational form. We are merely putting in the shading in our diagram.

We are now in a position to understand Boole's rule for Elimination. It is simply this.—Let $f(x) = 0$ be any logical equation involving the symbol x, then $f(1)f(0) = 0$ is the full expression for the result of the elimination of x from it.

This rule does not, at first sight, appear to have any connection with either of those offered in the last chapter, but a little consideration will show that it is substantially identical with the second of them. It only differs in fact by offering a methodical formula for a process which we had partly left to empirical judgment. We shall soon see this by examination of an example. Take the following,

$$w = xy + \bar{x}z,$$

and suppose we want to eliminate y from it. We have $f(1) = w - x - \bar{x}z, f(0) = w - \bar{x}z$, so that the rule gives us

$$(w - x - \bar{x}z)(w - \bar{x}z) = 0.$$

Multiplying out, this reduces to

$$w\bar{x}\bar{z} + \bar{w}\bar{x}z = 0,$$

or exactly the same result as we should have obtained by

breaking up the equation into its separate denials and selecting those only amongst them which do not involve y.

But our present object is not so much to show that the two methods agree in their result, as to explain the logical meaning underlying this abstract symbolic formula. This may readily be done as follows. We have shown that by eliminating y is meant the selection of those elements of denial which do not involve y or $\bar{y}$. We were then met by the difficulty that in the complete development *every* term must involve one or other of these. True, but those which are in effect explicitly free from y and from $\bar{y}$ are exactly those which involve *both* y and $\bar{y}$, for $w\bar{x}\bar{z} = w\bar{x}\bar{z}(y+\bar{y})$. Hence all that we have to do in the complete development is to select those elements only which involve both y and $\bar{y}$. Thus if the complete development of $f(y)$ is $Ay + B\bar{y}$, then the terms really free from y are those, and those only, which occur in both A and B. But the way of finding the elements common to both A and B is simply to multiply A by B. In other words, the product AB, that is, $f(1)f(0)$, is the symbolic expression for those elements of denial which the equation can yield free from y or $\bar{y}$. And the statement of such denial is given by equation to zero, so that $f(1)f(0) = 0$ is the precise symbolic statement of that process which we have worked out logically.

The same conclusion would follow as a simple corollary from the results of page 370. Thus the development of $f(y) = 0$ assuming the form $Ay + B\bar{y} = 0$, and these being mutual exclusives, each term must vanish separately. But the meaning of $Ay = 0$ is that none of class A is y, and the meaning of $B\bar{y} = 0$ is that all class B is y. Hence no A is B, or $AB = 0$; that is, as before, $f(1)f(0) = 0$[1]. (It will be

[1] In the first edition of this work I had said, in reference to a criticism by Ulrici upon the Boolian method, viz. "that it offered nothing new,

remembered that this result was only exceptionally true when $f(1)$ and $f(0)$ were derived from a simple class group instead of from an equation.) This expression being certainly free from y, and moreover all that can be obtained free from it, is a true elimination of y.

The case just hinted at, in which $f(1)$ and $f(0)$ happen to be exclusives, so that $f(1)f(0)$, being formally $=0$, leads to no result, deserves a moment's notice. It tells us that the term in question cannot be eliminated. Thus in

$$xyz + \bar{z} = y\bar{z} + x\bar{y}\bar{z},$$

we have, in trying to eliminate x, $f(1) \times f(0) = yz \times \bar{y}\bar{z} = 0$ from which nothing can be deduced. The meaning of this is that every term in the ultimate development involves x or $\bar{x}$ *separately*; so that we cannot free any term from its x condition by addition of two elements. But where every constituent element is thus conditioned it is clear that nothing can be inferred about the data when free from the condition. A more obvious case is afforded by the statement $xy = xz$. If we tried to eliminate x we should merely get $\frac{0}{0}y = \frac{0}{0}z$, from which nothing follows. The only assigned relation here between y and z being given under condition of x, no inference can be drawn as to what relations there may subsist between them in absence of x.

but was essentially nothing but a translation of the old Formal Logic into mathematical formulæ," that I should much like to see the original of this translation in the case of the elimination formula, $f(1)f(0)=0$. To this Mr C. J. Monro replied that "it is the conclusion of a syllogism in Celarent, Cesare, or Camestres, the term $f(1)$ standing in the negative premiss, and $f(0)$ in the affirmative" (*Mind*, VI. 580). Technically this is quite true, and is implied in the above paragraph in the text. In fact it has been one of my objects throughout this work to show that every so-called mathematical process here is at bottom logical. But it is the generalizations, and the power of direct application of our formulæ to groups of complicated propositions which put such a different face on the matter.

To the formula of Boole $f(x) = f(1)x + f(0)\bar{x}$, Prof. Peirce has added the companion formula

$$f(x) = (f(1) + \bar{x})(f(0) + x).$$

This follows at once from the rule for simplified multiplication given on p. 69, viz. $Ax + B\bar{x} = (A + \bar{x})(B + x)$. What these formulæ state is that any logical expression involving a class term x may be thrown into either of two forms. It may be developed as the sum of two products containing respectively an x and a $\bar{x}$ part, or as the product of two sums containing respectively the same parts. (See Peirce on *Algebra of Logic*. Amer. Journ. of Math., Vol. III.)

The symbolic extension for the formula of Elimination in the case of particulars, corresponding with that of generals, will be readily understood.

Let $f(x) = Ax + B\bar{x}$, be declared existent, *i.e.* in part or in whole.

Then we have

$$Ax + B\bar{x} > 0.$$

All that this asserts is the existence of something that is Ax or $B\bar{x}$. By the simple and immediate logical inference sometimes called 'inference by omitted determinants' we are therefore justified in saying (as was shown in the last chapter) that A or B exists; and in doing so we are making the elimination required. That is, we may express it,

$$f(1) + f(0) > 0.$$

The corresponding generalized expression for the elimination of a term from a pair of propositions, one universal and the other particular, may be expressed in the following formula. Given $f(x) = 0$, $\phi(x) > 0$, the elimination of x is contained in the formulæ,

$$\left.\begin{array}{r} f(1)f(0) = 0 \\ \phi(1).\overline{f(x)} + \phi(0).\overline{f(0)} > 0 \end{array}\right\}.$$

This is nothing more than the rule given in the last chapter (page 372) stated in more general symbolic language.

Another formula[1] may be stated thus,

"If everything is $f(x)$ and something is $\phi(x)$, the elimination of x is given by

$$\begin{cases}\text{Everything is } f(1) \text{ or } f(0),\\ \text{Something is } \phi(1)f(1) \text{ or } \phi(0)f(0).\text{"}\end{cases}$$

This can readily be shown to be equivalent to the former. Let $f(x) = Ax + B\bar{x} = 1$. Contradict $Ax + B\bar{x}$, and we have $\bar{A}x + \bar{B}\bar{x} = 0$. That is, the elimination of x yields $\bar{A}\bar{B} = 0$,

or $$A + B = 1; \text{ or } f(1) + f(0) = 1 \text{............} (1).$$

Again, let $\phi(x) = Cx + D\bar{x}$. Then, substituting in this expression the values of x and $\bar{x}$ as yielded by $\bar{A}x = 0$, $\bar{B}\bar{x} = 0$, we have

$$CA + DB > 0.$$

That is, $$\phi(1)f(1) + \phi(0)f(0) > 0 \text{..............} (2),$$

which are the two results required.

Boole generalizes this formula of elimination to embrace the simultaneous treatment of any number of terms. Thus the formula for eliminating the two terms x and y from $f(x, y) = 0$ is

$$f(1, 1)f(1, 0)f(0, 1)f(0, 0) = 0,$$

and so on, for any number of terms. The reader who has followed the explanation above will without difficulty see the logical interpretation here. For instance, when the expression $f(x, y)$ is fully developed, the only members of the series of constituent denials which can be exhibited free

[1] Due to Mr W. E. Johnson (Examination question in *Moral Sciences Tripos*, 1887).

from both x and y are those which simultaneously involve all the four elements xy, $x\bar{y}$, $\bar{x}y$, and $\bar{x}\bar{y}$ (for the sum of these $= 1$). Now the above formula of elimination is nothing but the logical rule for selecting such elements. And so on with three, four, or more simultaneous eliminations.

We may apply these formulæ in this shape; but, as was before remarked, the process of 1 and 0 substitution is very awkward and liable to error in practice. The student will therefore find it better to adopt the same plan as when only one term was in question; as follows. Collect all the elementary denials together, and arrange the sum of them in the form

$$Axy + Bx\bar{y} + C\bar{x}y + D\bar{x}\bar{y} = 0,$$

where A, B, C, D, are compounds of the other terms involved. Then the elimination of x and y is given by, $ABCD = 0$.

The explanation of this formula will readily be understood from what has been already said. One advantage in resorting to it consists in the fact that we may often thus detect at a glance that elimination is *not* possible, and therefore be saved the trouble of proceeding any further. If, for instance, *any pair* of the four, A, B, C, D, are mutually inconsistent, it is obvious that the whole product $ABCD$ will simply disappear, instead of yielding a result which we have to equate to zero. Equally so in case any one of these four terms is $= 0$, that is, does not appear at all. We then recognize at once that there can be no elimination of both x and y. Or again; if any one of these four terms $= 1$, then it may simply be left out of account, and only the other three be multiplied together. A glance, for instance, at the following expression would show that it failed on both the grounds mentioned above,

$$zw\,(u\bar{v} + \bar{u}v)\,xy + ce\,(f + g)\,\bar{x}y + \bar{z}fg\,(c + e)\,x\bar{y} = 0.$$

The reference to the peculiar symbol $\frac{0}{0}$ a few pages back affords a convenient opportunity for making some concluding remarks upon its significance. It is of course a logical factor, standing therefore for a logical term, but its peculiarity is that it stands for any term whatever and is therefore perfectly indefinite. We did not, it will be remembered, borrow it directly from mathematics, but found that it spontaneously presented itself in certain cases of the performance of the inverse operation denoted by the fractional sign. This, I think, suggests a caution or restriction in respect of its use. We ought not to regard it as in any way dividing any ultimate class subdivision to which it is prefixed. If we had invented it for ourselves, as a sort of substitute for the 'some' of ordinary logic, it might be asked why we should not prefix it at once to any simple class term, and therefore write down at our own prompting such an expression as $\frac{0}{0}x$, instead of waiting to incorporate it into an equation. The objection to so doing is, I think, connected with a fundamental characteristic of our whole scheme. Given a class term x, and this only, we have no right to talk of a *part* of x. A class, with us, is only divisible by its combination with another class term; that is, given y also, we can at once contemplate a division of x into xy and $x\bar{y}$, but we need the presence or suggestion of such another term in order to do this.

Herein lies the difficulty,—I should say the impossibility,—of representing the true particular proposition by aid of $\frac{0}{0}$ or any symbolic equivalent. From Boole's $vx = vy$, if v be fully indefinite, it is clear that nothing whatever can be obtained; for $\frac{0}{0}x = \frac{0}{0}y$ has not a definite word to say upon any subject. This expression therefore will not subserve the purposes of ordinary thought. As regards Jevons' $AB = AC$ the case is rather different. Of course if A were really indefinite (as he sometimes claims it to be) the two expressions

would be identical; but as he generally terms his form a 'partial' or 'limited identity', this seems to point to another signification. What we are supposed to express is the fact that 'the B which is A is the same as the C which is A': in other words, that, within the restricted sphere of A, B and C are identical. In this expression we must assume that A, though a common class term, is of unknown import, as otherwise we should merely have, not a particular, but a convertible universal of the type 'All X is all Y'. Is then $AB=AC$, where A is thus unknown, a fair type of the particular proposition? I think not, because it postulates that the members common to B and to C have enough attributes in common to be entitled to a connotative name, though we may not know what the connotation is. But what right have we to assume, because there are individuals common to B and to C, that therefore these individuals must possess such attributes in common? *One* such attribute I admit them to possess, viz. that of their common membership in B and C, but this would clearly only yield an identical proposition[1], and we have no warrant for insisting upon any other such attribute. Hence, although $AB=AC$ can be read off, as 'some B is C', it is not true that every 'some B is C' can be correctly formulated as $AB=AC$. Examples in point are common enough in science. Thus, of the honeysuckles in the English *Flora* those which climb have their flowers in whorls, and conversely. When therefore we omit the reference to England we can only say that '*some* climbing honeysuckles have their flowers in whorls'. But is this a fair type of an ordinary particular proposition? I think not. I can see no form which will cover all par-

[1] If A stands for the fact of belonging to both B and C, then $AB=AC$ becomes $BCB=BCC$, or $BC=BC$, which is clearly tautological;—as Spalding practically points out (*Logic*, p. 64).

ticular propositions, except that which throws them into an assertion of existence, and confines itself to declaring that there are B's which are C, as explained in the seventh chapter.

In his *Studies* (1880) Jevons gave another form, viz. $CA = CAB$, to represent the particular affirmative. It only differs from $CA = CB$ by saying that, within the sphere of C, 'A is B', instead of 'A and B are identical'. In so far as this goes it is clearly an improvement; but if C is to be regarded as a class term neither form can be considered as at all answering to the indefinitude of the true particular. Indeed on Jevons' own principle (viz. that the one criterion of necessity is that no single term is to be equated to zero; but that any combinations of them may be so equated) these forms will not answer his purpose. There is nothing in $CA = CB$ or $CA = CAB$ to prevent us from asserting that 'No A is B': the abolition of AB would still leave parts of A, B, and C undestroyed.

After what has been already said it hardly needs repeating that $\frac{0}{0}$ is not the equivalent of *some*. Probably the best compendious statement of its significance is that it is a confession of entire ignorance in respect of the term to which it is prefixed[1]. If it be asked why we want such a symbol, the answer has been already given in the results of the Chapter on Development: viz. that the comprehensiveness of our system, in which we frame a perfectly complete scheme of subdivision and call for an answer as regards every compart-

[1] Though therefore it is quite true to say that "either 1, 0, or $\frac{0}{0}$, multiplied into itself, equals itself" (*The Logic of Names, an introduction to Boole's Laws of Thought*, by I. P. Hughlings:—a little book with some good points in it, but which hardly fulfils its title), this should not be said without a caution. The two former multiplications really do result in known equality; in the third case what is meant is that our ignorance is equally complete before and after the multiplication.

ment contained in it, necessarily demands the equivalent of such a symbol. A confession of perfect ignorance in respect of a single class, taken by itself, has no determinate significance; but the same confession in reference to one or more remaining classes of an exhaustive catalogue, after we have definitely pronounced upon all the preceding classes, has a decided significance.

Thus $\frac{0}{0}x$, by itself, intimates nothing definite; and $\bar{x}y+\frac{0}{0}x$ intimates nothing definite in respect of its second term. But make these expressions members of a logical equation or statement, and the indefinite elements immediately acquire a certain significance. Thus $z=\frac{0}{0}x$ cannot tell us whether z is any actual part of x, but does assure us that it is no part of not-x, and this is important. Similarly $z=\bar{x}y+\frac{0}{0}x$ assures us that z is no part of not-$(\bar{x}y+x)$, which is decidedly different from informing us that it is no part of not-$\bar{x}y$. In every case the indeterminate term represents a confession of ignorance over its whole range, but to confine our ignorance within that range is to yield knowledge in reference to what is outside it, and this is done in any of these logical equations.

CHAPTER XVI.

COMPLETION OF THE LOGICAL PROBLEM. THE SYLLOGISM.

WE have now reached the last step of our purely logical analysis. We have shown how to resolve any propositions, and any groups of propositions, into all their ultimate denials, that is, into all the unconditional elements which they contain. This was the first step, and was fully treated in what was said about Equations and the Interpretation of Equations. The next step was to show what could be done with a portion of these elements; that is, how nearly we could retain the full force of the propositions in question by the use of a selection only of the total number of terms involved in them. This was treated in the Chapter on Elimination.

What we have now to do is to take a step in the way of Synthesis. We want to investigate some rule for determining the value of new groups of these elements in terms of the given class symbols. The full symbolic statement of the proposed step would be this;—Given $f_1(x,y,z,\ldots)$, $f_2(x,y,z,\ldots)$, &c., determine[1] $F(x, y, z,\ldots)$ in terms of any assigned selection of

[1] This F (x, y, *&c.*) may involve any selection, of course, of the terms x, y, &c. It is not only a new function of the class terms, but a new function of a *selection* of them.

the remaining symbols. A special case of this process may be detected in that generalization of the syllogistic process already referred to amongst the examples in Chap. XIII., p. 350, when we determined xz from xy and yz. That is, we have,—

$$\left.\begin{array}{ll} \text{for } f_1(x, y, z, a, c) = 0, & xy - a = 0, \\ \text{for } f_2(x, y, z, a, c) = 0, & yz - c = 0. \end{array}\right\}$$

The problem is, find $F(x, z)$, or xz, in terms of a and c, omitting y from the conclusion; *i.e.* eliminating y.

The general solution of this problem was probably first conceived, and almost certainly first effected, by Boole. As a piece of formal symbolic reasoning there seems nothing to be added to it as he left it, and it is a striking example of his penetration and power of generalization. It cannot often be the lot of any one to conceive and so completely to carry out such a generalization in an old and well-studied subject.

We will approach this problem in the same way as we have attacked the previous ones, that is, by first seeing what suggestions our common logical knowledge could offer towards the solution of it. We will then turn to Boole's method of solution; the real logical significance of which is by no means easy to grasp, unless we have thus examined the matter first in a somewhat more empirical way.

Take the simple example offered above: viz.

$$\text{Given } \left.\begin{array}{l} xy = a \\ yz = c \end{array}\right\}, \text{ find } xz \text{ in terms of } a \text{ and } c.$$

We should naturally begin by breaking up the given equations into all their ultimate denials, so as to obtain the whole materials for whatever they can affirm, deny, or leave in doubt.

These materials are the following:—

$$a\bar{x} = 0, \qquad c\bar{y} = 0,$$
$$a\bar{y} = 0, \qquad c\bar{z} = 0,$$
$$\bar{a}xy = 0, \qquad \bar{c}yz = 0.$$

Now develop xz, the quantity to be determined, in terms of the other elements, and we have

$$xz = xz\,(acy + ac\bar{y} + a\bar{c}y + a\bar{c}\bar{y} + \bar{a}cy + \bar{a}c\bar{y} + \bar{a}\bar{c}y + \bar{a}\bar{c}\bar{y})\ \ldots(1).$$

This is, the reader will remember, a merely formal result, an expression which must always hold good. We shall now proceed, so to say, to materialize it, that is, to bring it into relation with the assigned data, by seeing to what it reduces on the introduction of the above denials. Remove then from it all those elements which, in virtue of the given equations, can be shown to vanish, and it reduces to

$$xz = xz\,(acy + \bar{a}\bar{c}\bar{y}) \text{ or } xz = \tfrac{0}{0}\,(acy + \bar{a}\bar{c}\bar{y}),$$

or, in the form in which we want it, in which xz is expressed only in terms of a and c, it would stand

$$xz = \tfrac{0}{0}\,ac + \tfrac{0}{0}\,\bar{a}\bar{c} \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

It must be observed, here, that the factor $\frac{0}{0}$ is introduced by a double right. For one thing we want to eliminate y, and we know that the most direct way of doing this is simply to substitute $\frac{0}{0}$ for the term to be eliminated. Then again, anything standing in the form $X = XY$ is known to give an indefinite value of X, since this is one of the alternative forms for $X = \frac{0}{0}Y$. Hence even if y had not been eliminated we should have known that we had only got a result of the form

$$xz = \tfrac{0}{0}\,acy + \tfrac{0}{0}\,\bar{a}\bar{c}\bar{y}.$$

When equation (2) is read off into words, it stands,

"All xz is either both a and c, or neither a nor c".

This answer is quite correct so far as it goes, but it must be carefully observed that it does not go so far as it might. One side of the equation is plain enough, but not the other. We know, that is, that xz is confined to ac and $\bar{a}\bar{c}$, but we do not know whether it contains the whole of either or both of these terms. This represents the present state of our knowledge; with further knowledge we might ascertain that one or both of these factors $\frac{0}{0}$ must be converted into 1. In order to decide this point we should, if we continued the same plan, have to examine both the elements ac and $\bar{a}\bar{c}$ in terms of xz, in order to determine whether we could thus partially or entirely convert the equation. As it happens, ac is very easily determined. For, multiplying together the two original equations, we have at once $ac = xyz = \frac{0}{0}\,xz$.

Hence it is clear that the equation can be written

$$xz = ac + \tfrac{0}{0}\,\bar{a}\bar{c},$$

viz. that 'all ac is xz', as well as 'all xz is ac or $\bar{a}\bar{c}$'. Again as regards $\bar{a}\bar{c}$ we have

$$\bar{a}\bar{c} = (1 - xy)(1 - yz).$$

This is not expressible simply in terms of xz. Accordingly we cannot convert the second term in the expression for xz, but must leave the equation as it stands above, viz.

$$xz = ac + \tfrac{0}{0}\,\bar{a}\bar{c} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Let us, for further illustration, vary the example by making it a trifle more complicated. Let it be proposed to determine the expression $x\bar{z} + \bar{x}z$ from the same data as before: that is, let $F(x, z) = x\bar{z} + \bar{x}z$. Proceeding exactly as before, by developing each of the expressions $x\bar{z}$ and $\bar{x}z$ in respect of the remaining three terms, and omitting those

elements which the original equations prove to be non-existent, we have

$$x\bar{z} = x\bar{z}\,(a\bar{c}y + \bar{a}\bar{c}\bar{y}) = \tfrac{0}{0}\,a\bar{c} + \tfrac{0}{0}\,\bar{a}\bar{c},$$

$$\bar{x}z = \bar{x}z\,(\bar{a}cy + \bar{a}\bar{c}\bar{y}) = \tfrac{0}{0}\,\bar{a}c + \tfrac{0}{0}\,\bar{a}\bar{c}.$$

Gathering the two together, and remembering that the multiplication by 2, or any other factors, of a term affected by the indeterminate factor $\frac{0}{0}$, still leaves it indeterminate, we have

$$F(x, z) = x\bar{z} + \bar{x}z = \tfrac{0}{0}\,a\bar{c} + \tfrac{0}{0}\,\bar{a}c + \tfrac{0}{0}\,\bar{a}\bar{c}$$

which is the required answer.

Here, as in the last example, the answer is not quite complete. If it were also required to know the converse, or converses, of this equation; that is, to determine whether $x\bar{z} + \bar{x}z$ included the *whole* of any one or more of these classes $a\bar{c}$, $\bar{a}c$, $\bar{a}\bar{c}$, we should have to take each of them in turn, and ascertain whether either of the three was included in $x\bar{z} + \bar{x}z$. We might proceed to do this by the complete method of developing $a\bar{c}$, $\bar{a}c$, $\bar{a}\bar{c}$, and then striking out the terms which are proved to be non-existent. But as it happens there is a much shorter way, as follows:

We have $a = xy$, $c = yz$,

$$\therefore\ a\bar{c} = xy\,(1 - yz)$$

$$= xy - xyz = xy\,(1 - z) = xy\bar{z}$$

$$\therefore\ a\bar{c} = \tfrac{0}{0}\,x\bar{z}.$$

$$\text{Similarly } \bar{a}c = \tfrac{0}{0}\,\bar{x}z.$$

The remaining term $\bar{a}\bar{c}$ does not admit of such statement. Hence, finally, we have,

$$x\bar{z} + \bar{x}z = a\bar{c} + \bar{a}c + \tfrac{0}{0}\,\bar{a}\bar{c},$$

from which it appears that the proposition may be converted

as regards the two terms $a\bar{c}$ and $\bar{a}c$. It is fully stated in words by saying, "The classes represented by $x\bar{z}$ and $\bar{x}z$ are both certainly contained in the aggregate of the classes comprised by $a\bar{c}$, $\bar{a}c$, and $\bar{a}\bar{c}$; and conversely, both $a\bar{c}$ and $\bar{a}c$ are contained in the aggregate comprised by $x\bar{z}$ and $\bar{x}z$";—or, in somewhat more familiar language, 'Every $x\bar{z}$ and every $\bar{x}z$ is either $a\bar{c}$ or $\bar{a}c$ or $\bar{a}\bar{c}$; and conversely, every $a\bar{c}$ and every $\bar{a}c$ is either $x\bar{z}$ or $\bar{x}z$'.

The process above illustrated is a perfectly general one, and if nothing more were desired in a logical process than the solution of the assigned problem, it would probably be the best, that is, the most effective and convenient way of setting to work. It may be described as follows, in a series of perfectly intelligible logical steps:—Take the given equations and analyze them into all their constituent elements, that is, into all the ultimate denials which they involve and which collectively make up their significance. Then take the given function, of which we are told to find the value, and make the requisite synthesis. That is, build up successively each part of it, employing for this purpose the above-mentioned denials. This latter stage is really one of rejection, for we begin by developing the required function into its full complement of potential classes, and then strike out as many of these as are shown to vanish in consequence of the previous analysis. Having thus gone through the Analysis and the Synthesis there remains the third step, namely that of Elimination. It may be required to express the desired function in terms of part only of the terms involved in the equations. If so, the elimination is of that easy kind discussed in the earlier part of Chap. XIV., in which the terms to be eliminated entered on one side only of the explicit equation. Substitute therefore the indefinite symbol $\frac{0}{0}$ for the terms to be eliminated, and the whole problem is solved,

so far as the determination of the given function in terms of the assigned class symbols is concerned.

If we want also to determine how many converse statements can be made, that is, to determine not only the classes of which the given function is composed, but also in what cases it comprises the *whole* of these classes, then we must go through the same processes in the case of each of these classes. We must take each of them in turn and build it up in the same way as the given function was built up. The same list of elementary denials will serve, of course, in each case, for we are dealing with one and the same set of original equations.

In familiar language the process may be described by saying that we take the given premises, break them up into fragments, and then put these fragments, or a part of them, together in some other arrangement in order to build up the structure we require.

Boole's plan for attaining this end is one which would probably seem the most natural to any mathematician who was disposed to apply to Logic the methods found so successful in his own science. He takes the assigned function $F(x, y, z)$ and puts it equal to, say, t; where t is of course simply a new symbol, the equivalent for this function. Our equations then stand thus:—

$$f_1(x, y, z) = 0,$$
$$f_2(x, y, z) = 0,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$F(x, y, z) - t = 0.$$

Now eliminate from these equations, after reduction by the methods described in a former chapter, every term except t and those terms in which $F(x, y, z)$ was to be determined.

Then develop t, by the well-known methods, and what was required is done; for t or $F(x, y, z)$ will be described in terms of those symbols, and those symbols only, which it was desired to make use of in describing it.

If certainty and completeness of symbolic procedure were all we had to look for, there can be no doubt that Boole's method would succeed, as the whole answer is at once and completely given by it; that is, we obtain at one and the same time the converse propositions referred to a page or two back, as well as the direct proposition describing $F(x, y, z)$. But it is a terribly long process, as any one will see who works through his solution, by this method, of such a simple problem as that on p. 350, viz.

$$\text{Given } \left.\begin{aligned} xy &= a \\ yz &= c \end{aligned}\right\} \text{ find } xz.$$

Pages of work are demanded in order to complete this comparatively simple problem.

THE SYLLOGISM.

We must frankly remark that from our point of view we do not greatly care for this venerable structure, highly useful though it be for purposes of elementary training in thought and expression, and almost perfect as it technically is when regarded from its own standing point. But its ways of thinking are not ours, and it obeys rules to which we own no allegiance. To it the distinction between subject and predicate is essential, to us this is about as important as the difference between the two ends of a ruler which one may hold either way at will. To it the position of the middle term is consequently worth founding a distinction upon, to

us this is as insignificant as is the order in which one adds up the figures in an addition sum. On the other hand the distinction between universal and particular propositions which to it is unimportant is to us vital.

There are reasons nevertheless for taking some account of the syllogism here; partly because the contrast of treatment will serve to emphasize this difference in the point of view, partly because the omission of any such reference might possibly be taken as a confession of failure on the part of the Symbolic Logic. Since the Syllogism is a sound process it must admit of some kind of treatment upon any scheme.

There are two ways of treating it. The method which would naturally be adopted by any one familiar with the use of symbols, but entirely ignorant of logical tradition, would probably be this. He would begin by rejecting all distinctions of Figure, as utterly alien to his scheme; and, as the common System admits that the other three figures can be reduced to the first, he would insist upon this simplification being made before he took the work in hand. That is, he would take account only of the first four moods. Then he would go on to reduce these by the consideration that, to his thinking, x and $\bar{x}$ being both classes of the same essential character, there was no occasion to formulate a distinction between moods which involved a negative, and those which contained only affirmative premises. There would then remain only the distinction between a form which draws a universal, and one which draws a particular conclusion. Even this he would, I imagine, bridge over by the consideration that, when we are concerned with the first Figure, the 'Some X' which occurs in the minor premise and in the conclusion is the *same* 'some' throughout, so that its indefinitude has no bearing whatever on the actual

process of reasoning[1]. Accordingly his syllogisms would all be rendered in one common form

$$\begin{array}{lcl} Y\bar{Z} = 0, & & Y = \tfrac{0}{0}\, Z, \\ X\bar{Y} = 0, & \text{or} & X = \tfrac{0}{0}\, Y, \\ \therefore X\bar{Z} = 0\,; & & \therefore X = \tfrac{0}{0}\, Z. \end{array}$$

When all the three letters stand for whole terms, these being positive, we have *Barbara.* When X stands for a part-class (*i.e.* for some-X), and Z is a negative term, we have *Ferio*; and so on with the two remaining forms. That is, the above symbolic arrangement will, by suitable interpretations of X, Y, and Z, cover all the Moods of the first figure, and will therefore indirectly meet the case of all the Moods of the other figures.

The above rendering of the syllogism, it will be seen, is really nothing but a symbolic translation of the *Dictum* of Aristotle; as any single comprehensive rendering of it ought, I suppose, to be. Or rather it is a slight generalization of that *Dictum,* for, since we recognize no difference of character between x and $\bar{x}$, we make but one *dictum* whether we say *omne* or *nullum.*

The above is one way of rendering the Syllogism; but

[1] Of course this is not the appropriate form of rendering particular propositions, as was fully shown in Chap. VII. But the problem here is not the same. What we then investigated was the best symbolic expression for such propositions, regard being had to the attendant difficulties of implication. What is now before us is the best way of expressing a certain process of reasoning. When a symbolist is driven to syllogize it is quite fair for him to render *Ferio* thus:—

All Y is not-Z:
Some-X is Y:
$\therefore$ Some-X is not-Z.

(The 'some-X' being consciously realized as the same 'some', it stands in essentially the same position here as an ordinary class term.) He does not syllogize willingly, nor claim to do it gracefully, but he can do it without actual error.

there is another way which will in some respects be more instructive, as it calls attention to the peculiarities of our interpretation of propositions. Put each of the 19 moods into symbols or diagrams as it stands, without previous reduction to the First Figure; and let Z, Y, X, be respectively the major, middle, and minor terms. We shall find these moods fall into three distinct classes.

(1) The first class comprises those which deal only with universal propositions. There are three of these; viz. Barbara; Celarent and Cesare (here indistinguishable); and Camenes and Camestres (also indistinguishable). Each premise destroys a combination of the middle term and one extreme, and the conclusion states the consequent destruction of a combination of the two extremes: $X\bar{Z}$ in Barbara, and XZ in the others. Thus in Celarent we have $YZ=0$, $X\bar{Y}=0$; $\therefore XZ=0$.

(2) The second class comprises those which deal with a universal and particular as premises, and have a particular conclusion. There are five of these: viz. Ferio, Festino, Ferison, and Fresison (indistinguishable); Darii and Datisi (indistinguishable); Baroco; Disamis and Dimaris (indistinguishable); and Bokardo. The peculiarity here is that one premise destroys a combination of the middle term and an extreme, whilst the other saves a combination of the middle and the other extreme; the conclusion in consequence saving a combination of the two extremes; XZ or $X\bar{Z}$ as the case may be. Thus Festino is rendered; $ZY=0$, $XY>0$; $\therefore X\bar{Z}>0$. That is, the saving of *some* part of XY (when ZY is gone) necessarily saves the $X\bar{Z}Y$ part; whence 'Some X is not Z'.

(3) The third class stands on a very different footing. In each of the moods; Darapti, Felapton, Bramantip, Fesapo, we have two universal premises with a particular conclusion.

Now it is impossible that the mere fact of destroying two classes (except in the familiar extreme case) should result in the *saving* of some other class. The inference really rests here upon that tacit assumption of the existence of things corresponding to the subject or predicate which we have had repeated occasion to notice. Thus (Darapti) from 'All Y is Z', 'All Y is X', we can only infer that 'Some X is Z' by help of the assumption that there *is* Y. As this assumption is no part of the universal, on the Symbolic interpretation, we have to introduce it explicitly if we want to draw the ordinary conclusion. If $Y\bar{Z} = 0$, $Y\bar{X} = 0$, *and there is* Y, this surviving part must be the YXZ: that is, Some X is Z. Similarly with the other three cases.

It need hardly be pointed out how widely this arrangement differs from that of the familiar four Figures, in respect both of the distinctions which we reject and of those which we introduce. (Substantially the same explanation is adopted by Miss Ladd in the Johns Hopkins *Studies*: see also Schröder, II. 217.)

Supposing that we feel bound to treat the syllogism at all, one of the above two certainly seems to me the best way of doing so; indeed the only way, in strict consistency with our own principles. One serious defect, as it seems to me, in the great majority of the attempts to treat Logic symbolically has consisted in the fact that the authors have not sufficiently shaken themselves free from the old trammels. They have felt bound to adhere as far as possible to all the old distinctions in the form, order, and so forth, of the constituent propositions even of the syllogism. The majority of the older symbolists (for instance, Maimon) have really done little more than go in detail through the nineteen moods, clothing each in a new symbolic dress. This is highly unsatisfactory, since most sets of symbols require some

violence to force them into recognizing distinctions so utterly alien to their genius and habits.

Boole's plan is very different. He has certainly solved a very general problem, and one which can be made to include, amongst other things, a number of the syllogistic moods. I cannot however regard it as a fair generalization of that process; nor, for that matter, did he seem to regard it so himself. His plan is as follows. He considers the two propositional forms, $vx = v'y$, $wz = w'y$, as containing under them all the possible forms of premise needed for a syllogism. Thus, put $v = 1$ in $vx = v'y$, and we have 'All x is y'; leave $v = 1$, but regard y as negative, and we have 'No x is y'; make v and v' indefinite, and we have 'Some x is y'; do the same, regarding y as negative, and we have 'Some x is not y'. Similarly with the interpretation of the other premise. Now eliminate y from these two equations and determine the relation of x to z, and a part[1] of the syllogistic scheme will be completed.

[1] Only a *part*, on two accounts. Firstly, in the propositions as above expressed, the same middle term y occurs in both. But, in the syllogism, we require that y shall be combined with $\bar{y}$ as a middle term; that is, we must take account of the pair of equations $vx = v'y$, $wz = w'\bar{y}$, or we shall omit some of the recognized pairs of premises. Secondly, syllogistic conclusion demands the determination not only of the relation of x, but also of that of $\bar{x}$ and vx, to z, or we shall omit some of the recognized conclusions. Consequently the form above worked out is only one of six which demand examination, and which are duly discussed by Boole.

It deserves notice that Lambert attempted much the same problem. Starting from two perfectly general expressions for the premises, he obtains a similar one for the conclusion, pointing out that by due determination of the arbitrary letters involved we may specialize for any desired figure and mood. His preliminary forms (of which some account is given in Chap. XX.) are closely analogous to those of Boole. Thus,

The answer obtained by Boole from these premises, involving seven terms, is somewhat intricate. It is given in the *Laws of Thought* (p. 232), and will be discussed immediately under its more appropriate interpretation. I will not however pursue the subject of the syllogism itself further, for two reasons. In the first place, the whole enquiry seems to me to be carried on upon a wrong line of attack, inasmuch as it involves a concession to a variety of rules and assumptions which from our point of view must be regarded as arbitrary and almost unmeaning. Moreover, as has been abundantly shown, such a form as $vx = v'y$ cannot be regarded as a true representation of a particular proposition unless we reject the value 0 for v and v', a restriction which is not claimed for x and y.

If we simply regard v, v', w, w', as ordinary class terms, like x, y, z, the problem in question acquires a very ready interpretation, but one widely remote from anything contemplated in syllogistic Logic. It then becomes, "If every x which is v is a y which is v' (and *vice versâ*), and every z which is w is a y which is w' (and *vice versâ*), what is the description of x as given by the other terms, omitting y"? A concrete example of this is given by Boole himself in the words:—"Suppose a number of pieces of cloth striped with different colours were submitted to inspection, and that the two following observations were made upon them;

(1) That every piece striped with white and green was also striped with black and yellow, and *vice versâ*.

"Major $\dfrac{mA}{p} = \dfrac{nB}{q}$,

Minor $\dfrac{\mu C}{\pi} = \dfrac{\nu B}{\rho}$,

Conclusio $\dfrac{\mu n}{\pi q} C = \dfrac{m\nu}{p\rho} A$.

In dieser allgemeinen Formel kann man die Buchstaben nach Belieben bestimmen, wenn man daraus besondere Formeln für die Schlüsse herleiten will" (*Log. Alg.* [illegible] 103).

(2) That every piece striped with red and orange was also striped with blue and yellow, and *vice versâ*.

Suppose it then required to determine how the pieces marked with green stood affected with reference to the colours white, black, red, orange, and blue".

As v' and w' are not convenient letters for printing with a bar over them, we will substitute others, and write these two premises in the form, $ax = cy$, $ez = vy$.

There are a variety of ways of solving this problem. I purposely choose an unusual way in order to compare the result with those obtained in other ways.

Eliminate then y, by equating the two values assigned to it, and we have

$$ax + \tfrac{0}{0}\,\bar{c} = ez + \tfrac{0}{0}\,\bar{v}.$$

This yields the denials, $ax \cdot \overline{ez + \bar{v}} = 0$, $ez \cdot \overline{ax + \bar{c}} = 0$; or

$$\left.\begin{array}{l} avx\,(\bar{e} + \bar{z}) = 0 \\ ecz\,(\bar{a} + \bar{x}) = 0 \end{array}\right\};$$

add to these the usual conditions of fractional interpretability, $ax\bar{c} = 0$, $ez\bar{v} = 0$, and we have, as the full significance of the premises (by the formula on page 313),

$$x\,(av\bar{e} + av\bar{z} + a\bar{c}) + \bar{x}cez + \bar{a}cez + e\bar{v}z = 0.$$

This yields, as the complete value of x,

$$cez + \tfrac{0}{0}\,(\bar{a} + \bar{v} + e)\,(\bar{a} + \bar{v} + z)\,(\bar{a} + c)\,(\bar{c} + \bar{e} + \bar{z}).$$

Multiply out, and we have

$$x = cez + \tfrac{0}{0}\,(\bar{a} + c\bar{v})\,\bar{z} + \tfrac{0}{0}\,(\bar{a}\bar{c} + \bar{a}\bar{e} + c\bar{v}\bar{e}) \;\ldots\ldots (1).$$

Or, in words, 'Every piece striped with green is striped with black, red, and orange; or with neither white nor orange; or with black but neither blue nor orange; or with neither white nor black; or with neither white nor red, or with

black and neither red nor blue. And every piece striped with black, red, and orange is striped with green'.

The value of x which Boole gives is somewhat different, viz.

$$acevz + \tfrac{0}{0}(ac\bar{e}\bar{v} + \bar{a}\bar{c}ev + \bar{a}\bar{e})z + \tfrac{0}{0}(ac\bar{v} + \bar{a})\bar{z} \ldots\ldots (2).$$

Professor Schröder gives a briefer answer[1], viz.

$$cez + \tfrac{0}{0}(\bar{a} + c\bar{v}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

It must be observed that all these values of x are different, as they stand. This is not a case, such as we have had before, of *equivalent* renderings which can be translated into formal identity. The symbolic expressions to which x is respectively equated cannot, by any legitimate transformation, be equated to each other. They all cover, in great part, the same range, but some extend further than others.

The obvious reply is that these renderings are *not* to be taken as they stand. There are premises behind them, which determine their range of application, and in fact reduce them all to equivalence. We may illustrate in this way. Take a diagram such as that described on p. 140. If we were to mark on it all the ultimate subdivisions comprised by each of these three expressions (say, by vertical, horizontal, and inclined lines respectively) we should find the results different. But if we prepare the diagram first, by erasing the compartments destroyed by the premises, then we should find all these values reduced to identity. That is, though the symbolic or potential range of these expressions is different, their material or actual value under the circumstances of the case, is precisely the same.

[1] I give it, slightly modified, in the same symbols employed for the other cases. Of course, under the conditions involved, other forms might be proposed besides the above three.

This fact explains the discrepancy, but it leaves a question to be answered. An objection has been raised to my solution,—and this applies à fortiori to the still briefer solution,—that, as compared with Boole's result, it is deficient in that it demands an appeal to the premises for its complete justification. This comparison does not seem to me to have much force as regards the first, or definite term in the answer. Boole's solution says, when put into ordinary propositions, that "All x is either *acevz* or (other terms)", and conversely "All *acevz* is x". Now, though in this case *cez* and *acevz* are actually the same, yet formally or symbolically the former is the wider. If therefore another solution yields the corresponding converse "All *cez* is x", I do not see how it can be denied that this offers here the wider or completer proposition, except by falling back upon the explanation that we know, by the premises, that *cez* and *acevz* are actually the same. The answer that would be the completest, in its mere form, would yield the two propositions, "All x is *acevz* or &c.", and "All *cez* is x".

As regards however the indefinite terms Boole's result is certainly the most precise. As these *include* the range of x, the narrower they can be made the better. For instance, in the $\bar{a}$ compartments of x, which are all thrown into one in Schröder's form, Boole, by the more elaborate expression $(\bar{a}\bar{z}+\bar{a}\bar{e}z+\bar{a}\bar{c}evz)$ confines x to 26 out of the 32 ultimate subdivisions. So with the a compartments: Schröder's form refers x to 8 of the subdivisions, Boole's confines it to 6 of these. Materially, as above stated, the two renderings are exactly equivalent, but formally the latter is the narrower and more precise.

The result I have myself given above is intermediate in this respect, at least as it there stands. It refers x to 28 out of the 32 $\bar{a}$ subdivisions, and to 6 out of the a

subdivisions: agreeing in this latter respect with Boole's answer. It must however be remembered that when we employ a fractional expression we are supposed to have the condition of interpretability at hand: this is a part of our conclusion, and to make use of it is not, I think, to be considered as an "appeal to the data", whether this be objectionable or not. If this condition be introduced as a correction of the first result we are brought at once into accordance (so far as the indefinite terms are concerned) with Boole's result. The conditions are; $ax\bar{c}=0$, $e\bar{v}z=0$. The former of these evidently cannot affect any of the indefinite terms, the latter can only be applied to the $\bar{a}\bar{c}$ constituent. Deduct then $e\bar{v}z$ from $\bar{a}\bar{c}$, by substituting $\bar{a}\bar{c}(\bar{e}+v+\bar{z})$ for $\bar{a}\bar{c}$, and the expression becomes,

$$x = cez + \tfrac{0}{0}(\bar{a}+c\bar{v})\bar{z} + \tfrac{0}{0}(\bar{a}\bar{e}+\bar{a}\bar{c}v+c\bar{e}\bar{v})$$

which, except for the wider (symbolic) assignment of the definite term, cez, accords precisely with Boole's result.

As regards the abbreviated form $x = cez + \tfrac{0}{0}(\bar{a}+c\bar{v})$ this can very readily be obtained. We have $vy = ez$, therefore $y = \tfrac{0}{0}(ez+\bar{v})$. Substitute in the other premise, and we have $ax = \tfrac{0}{0}(cez+c\bar{v})$, or $x = \tfrac{0}{0}(cez+\bar{a}+c\bar{v})$. All that then remains is to take the right-hand terms in order, and to see which of them can be rendered definite. Since $cez = cvy = avx$, it is plain that 'all cez is x'. Also, since $ax = c$, $x = \tfrac{0}{0}(c+\bar{a})$; viz. '$x$ is c or $\bar{a}$'. But the premises give us no right to assume that $c=0$, or $\bar{a}=0$, so we cannot say that either 'All c is x', or that 'All $\bar{a}$ is x'. Therefore the expression stands

$$x = cez + \tfrac{0}{0}(\bar{a}+c\bar{v}).$$

If the question be asked which of the two distinct forms is intrinsically the best, it does not seem to me that there is any certain canon by which to decide. Of course we may

so frame the problem as to demand the former, as by saying, 'Given certain premises involving x, y, &c.; determine x in the fullest possible form without employment of y'. But is it necessary that we should do this? For most purposes, whether speculative or practical, it is sufficient to know that our answer is complete under the conditions of the problem: that no case is omitted which the data render possible.

The subject is connected with one which has already occupied our attention. Jevons, it will be remembered, objected to the syllogistic conclusion 'X is Z' from the premises, 'Y is Z, X is Y'; on the ground that it did not contain the full information in the premises; and preferred his own form X is YZ. The answer we gave (page 363) was that the syllogism did not propose to do this, and that there did not seem any reason why, as a general rule, we should propose to do so.

CHAPTER XVII.

GENERALIZATIONS OF THE COMMON LOGIC.

THROUGHOUT this work we have been occupied at almost every step in considering the extensions which can be effected in the processes and results of ordinary Logic; but it will be convenient at this point, having now finished the systematic exposition of the subject, to gather together these various generalizations so as to see what sort of an appearance they present in the aggregate. This is the more necessary since their importance consists not merely in the fact of their being generalizations, but also in the decided change of view which the admission of some of them would demand; as, for instance, in respect of the Import of Propositions. Those points which have been adequately discussed in the foregoing chapters will only demand a very brief notice here.

To begin then: In place of the old dichotomy we have substituted a system of polytomy; that is, we divide the universe of things into all the ultimate subdivisions attainable by taking every term and its contradictory into account. We start at once with all these subdivisions as our normal

requirement, instead of regarding them as merely a possible attainment[1].

The point here which seems most characteristic of our system is the following:—That we put every one of the classes thus obtained, whether positive or negative, upon exactly the same footing. With us, xy and $\bar{x}\bar{y}$ are classes of precisely the same kind. We do not indeed strictly speak of positive or negative terms at all, regarding these words as grammatical or conventional expressions founded upon the popular necessities of speech and classification. Any compound term, such as $x\bar{y}z$, will commonly contain both positive and negative elements, and therefore these words cease to be appropriate to the compound. We have no reason indeed to assume that x, rather than $\bar{x}$, should represent what popular language would regard as a positive term. We regard x and $\bar{x}$, of course, as contradictories, but we almost decline to call either of them positive or negative.

This view of the primary indifference of application of x and $\bar{x}$, as such, is very alien to popular ways of thinking. To most persons the interpretation of concrete problems which turn upon the employment of negative classes is not a little

[1] The writer (before Boole) who has most strongly insisted on this enumeration of all the possible combinations produced by all the class terms, is Semler (*Versuch über die combinatorische Methode*, 1811). The empirical method which he proposes, viz. that of writing down all the combinations, and then scratching out or removing, individually or in groups, all those elements which are contradicted by the data, is almost identical with that of Jevons. "Setzen wir zum Beispiel man hätte die 56 Conternationen, die sich aus den Begriffen *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, bilden lassen, wohlgeordnet aufgeschrieben, und man bemerkte die Begriffe *a* und *b*, desgleichen *b* und *d* könnten ohne Widerspruch nicht zusammen gedacht werden, so kostete es auch nur ein Paar Federstriche, um die Columnen der 10 Conternationen wo die Zeichen dieser Begriffe beisammen stünden, wegzuschaffen" (p. 48).

perplexing. To borrow one or two of those apt and ingenious examples with which all De Morgan's logical writings abound, how many persons would be able to say confidently and off-hand whether either, and if so which, of these two statements is true? (1) The English who do not take snuff are included in the Europeans who do not take tobacco. (2) The English who do not take tobacco are included in the Europeans who do not take snuff. (Snuff-takers of course being included in tobacco-takers.)—Or, again, Who are the non-ancestors of all the non-descendants of $A.B$? (*Camb. Phil. Tr.* x. 334.)

When this view is taken it is seen at once that we must insist upon some reconsideration, for our special purposes, of the old logical account of contrariety and contradiction, both as regards classes and propositions.

First as regards *Classes*. The 'contradictory' of any class is best interpreted in its ordinary sense, viz. as comprising all which does not belong to that class. The points characteristic of our Logic are mainly these: that we may apply the same notation to contradict any class however complicated, and that we keep prominently in view all those elements of the full development into which the contradiction of any such class can be broken up. As regards the notation, what we require is either some symbol for the universe of discourse from which the given class has to be excepted, or some convenient symbol which can be applied to any class as a whole. The notation we have adopted meets both these requirements. Thus, if we want to contradict $x\bar{y}z + xy\bar{z}$ all we have to do is to write it $1-(x\bar{y}z+xy\bar{z})$; or, more briefly, draw a single line over the whole expression, just as we write $\bar{x}$ for $1-x$. Other methods of representing the same process will be found described in the concluding chap-

ter, but it seems to me a serious drawback[1] in any symbolic system if it does not meet these requirements.

The extended signification of the word *contrary* does not however seem by any means so clear and simple. In fact the word is so bound up with the traditional restrictions of the ordinary Logic that we shall have to make a considerable departure from customary usage if we want to assign a consistent signification to it. Perhaps the best account we could give (founded as far as possible on the customary usage) would be the following:—that as the contradictory of a given class is the sum total of what is exclusive of it, so the contrary of the given class is that class, within all this contradictory region, which is farthest removed from it. At least I presume that this is most in accordance with common usage. If we adopt this account we should say that such a class as xyz, involving three terms, has an aggregate contradictory comprising seven classes; and out of these seven we should select as the 'contrary' to it that one which contradicts it in every detail, in other words the class in which every constituent element is a contradictory of the corresponding element of the aforesaid class. Thus $\bar{x}\bar{y}\bar{z}$ would be the contrary of xyz, $\bar{x}y\bar{z}$ of $x\bar{y}z$, and so on.

If we had often occasion to speak of relations of this kind it would be better to abandon the term 'contrary' altogether, and to use instead some such expression as that of two class terms being contradictory in one, two, three, degrees, &c., and congruent in respect of the remaining, according to

[1] This is the case with Jevons' system, in which neither of these requirements is met. In his subsequently published *Studies in Deductive Logic*, he adopted Mr McColl's notation for the purpose.

In speaking above of the *representation* of contradictories, I need hardly say that we are not referring to the practical methods for working them out. These methods were discussed on page 280.

the number of constituent elements. We do not really find much occasion to employ technical expressions of this kind, but if we did, the best way of describing the relations in this respect of two such classes as $ab\bar{c}de$ and $\bar{a}bcd\bar{e}$ would be by saying that they are contradictory in the third degree and congruent in the second degree. But in saying this it must be remembered that the two classes in question are as entirely exclusive of each other as if the contradiction were more complete, for the state of exclusion of one class by another is absolute and does not admit of degrees. When we thus came to deal with classes not fully subdivided, as for example xy and xz, we should have to analyse them fully and then describe the degree of contradiction of their constituent elements separately. Thus xy and xz resolve respectively into $xyz + xy\bar{z}$, and $xyz + x\bar{y}z$. Of these the coincident, or doubly repeated element xyz, furnishes no contradiction; the remaining elements $xy\bar{z}$ and $x\bar{y}z$ would have a two-fold contradiction[1].

The generalized expressions for contradiction have been noticed from time to time in the preceding chapters, but it will be convenient here to repeat some of them in their widest form[2].

[1] The late W. K. Clifford proposed a scheme of nomenclature for propositions as grouped from this point of view, but it is too intimately connected with the particular discussion with which he was there occupied to be reproduced here. That discussion was alluded to in Chap. XII. p. 324. It is published in his Essays, and also an abstract of it in Jevons' *Principles of Science* (p. 143).

[2] This symmetrical form of contradiction was probably first proposed by De Morgan (*Camb. Phil. Trans.*, 1858, p. 208), "Thus (A, B) and AB have (ab) and (a, b) for contraries":—he denotes our $a+b$ by a, b. It deserves notice that, when he translates these expressions into mathematical symbols, he adopts the exclusive notation, "And (A, B) is not $A+B$, but $A+B-AB$; as pointed out by Mr Boole" (p. 185).

(1) To contradict a product of terms, add together the contradictories of all the separate terms,

$$\overline{xyzw\ldots} = \bar{x} + \bar{y} + \bar{z} + \bar{w} + \ldots.$$

This, of course, is on the non-exclusive notation: if we wish to keep the terms free from each other we must, with Boole, adopt the longer expression,

$$\bar{x} + x\bar{y} + xy\bar{z} + xyz\bar{w} + \ldots.$$

(2) To contradict a sum of terms, multiply together the contradictories of all the separate terms,

$$\overline{x + y + z + w + \ldots} = \bar{x}\bar{y}\bar{z}\bar{w}\ldots.$$

(3) As an obvious extension we have, by combining the two previous rules, the direction for contradicting any expression whatever:—Change every single term into its contradictory; and change every sign of multiplication into the sign of addition, and *vice versâ*.

Thus

$$\overline{x\bar{y}z + \bar{x}yz + xy\bar{z}} = (\bar{x} + y + \bar{z})(x + \bar{y} + \bar{z})(\bar{x} + \bar{y} + z).$$

Propositions. As regards the application of the terms contrary and contradictory to propositions, a moment's consideration will show us how far we have here wandered from the customary view. So much is this the case that it really seems impossible to assign to the terms any rational signification which shall remain in harmony with their customary meaning. Thus for instance the two propositions, $xy = 0$, $x\bar{y} = 0$, represent respectively 'no x is y', and 'all x is y'; and on the predicative view of propositions they certainly involve between them a contradiction, indeed a contrariety. But on the compartment or occupation view they simply clear out between them the whole of x, one accounting for the y-part of it and the other for the not-y part of it. On

this view therefore they would much more truly be described as supplementary than as contradictory.

If we were resolved to find a use of the term which should be in harmony with our system we should best seek it in the following way:—We know that $xy=0$ denies that there is such a class as xy; that $xy>0$ (on the usual mathematical interpretation of the symbol adopted in Ch. VII. p. 185) asserts that there is such a class; and that $xy=1$ asserts the existence of that class to the exclusion of all else. We might therefore lay it down, with approximate conformity to common usage, that the first of these forms is contradictory of the second, and that the two extreme forms, $xy=0$ and $xy=1$, form a pair of contraries.

Such an interpretation as that last suggested is of course rather a shifting of the meaning of the terms in question than a generalization of them. Hardly any better instance could be found of the difficulty, in fact impossibility, of fitting in the details of the old system into the fabric of the new, or of the consequent necessity of reconsidering all the technical terms which we have occasion to employ. The value of these particular technical terms seems to me considerable in the old Logic, since they compel the student to realize the precise force and scope of such propositions as he is every day in the habit of using. But on our system there is but little opening of this kind for them. There would have been no use in discussing them here but for the fact that every effort to generalize the signification of well-known terms is a good mental exercise, and still more the attempt to realize so thoroughly the spirit and methods of different systems as to determine whether the technical terms of one system can be transferred at all, even in an extended sense, to another.

So much for the Opposition of Propositions: now turn to

their Conversion. Here again we are soon reminded how far we have wandered from the customary mode of treating this question. The predicative explanation of propositions, founded, as of course it is, upon a real distinction between the subject and the predicate, naturally leads to a demand for rules for the process of converting subjects into predicates and *vice versâ*; whereas on the compartment view of propositions we really have to stop and think whether any interpretation can be applied to the accepted rules for such a process. For instance, the conversion of a Universal Negative, $xy = 0$, can imply nothing more with us than an optional difference in the reading off of the proposition; we may phrase it as we please, 'No x is y' or 'No y is x'. And since we do not recognize any real distinction between positive and negative terms the same may be said of any other form. Given $x\bar{y} = 0$ we may phrase it indifferently, 'All x is y', 'No x is not-y', 'No not-y is x', or 'All not-y is not-x'. With us these are rather conversational than logical distinctions.

It seems therefore that in this direction we should not be led to a distinction of any value. But there is another way of regarding the process which is by no means synonymous with reading off one and the same statement with the other end foremost, as in common Conversion. From the statement 'all x is y' we commonly obtain 'some y is x'. When we look at this in the light of our symbolic system we see that it may be regarded as a very special case of this general problem[1]:—Given x as a function of y, find y as a function of x.

[1] It should be noticed that we must not speak of *the* generalization of a process, as if there could be but one. A process carried out (as must always be the case) under a variety of restrictions, will necessarily lead to a variety of generalizations, according to the number of such restrictions removed, and the extent to which we remove them.

If we take this view of the process to be performed:—it would be almost absurd to continue to use the old term 'conversion' in reference to it:—we should describe it as follows. Let x be a logical expression, involving y amongst other terms, so that we may exhibit it in the form $x = f(y)$. It is desired to 'convert' this relation into one which shall explicitly exhibit y in terms of x. We indicate the desired expression, in accordance with familiar mathematical usage, in the form $y = f^{-1}(x)$. The reader must not for a moment suppose that this solves the problem; or even implies that the problem admits of solution, whether determinately or indeterminately. The expression is rather a definition of what we want, $f^{-1}(x)$ being definable as anything which when acted upon by the process f will yield x; in other words, $f\{f^{-1}(x)\} = x$. This, of course, is only a generalization of a procedure with which the reader became familiar in the third chapter. Just as $\frac{x}{y}$ indicated any class which when logically restricted by y will yield x, so does $f^{-1}(x)$ indicate any class which when operated on by the general logical procedure represented by f, will yield x.

As it happens, the desired result can not merely be symbolically indicated, but is one which admits of actual performance; though here, as in most other cases of inverse procedure, the result is usually indefinite within certain limits. For, beginning with $f(y)$, this must be either a directly, or an inversely assignable logical class, these being the only logical functions which we recognize as data or can deduce as results. In other words $f(y)$ must be one of our integral or fractional forms, and, if a fractional one, is reducible to an integral form. Hence x, or $f(y)$, can be exhibited as the sum of a number of logical class elements, these being of course in the first degree. Accordingly, con-

versely, y can be exhibited (by logical processes corresponding to the solution of a simple equation of the first degree) as a function of x. That is, given $x = f(y)$, $y = f^{-1}(x)$ can always be solved and the result exhibited in a strictly intelligible logical shape. And we may call this a generalization of the ordinary process of Conversion.

I need not remind the reader that this is a process which we have frequently had to perform in the course of this work. Suppose, for example, that we had the relation $x = 1 - yz$, and were told to determine y in terms of x. We have

$$y = \frac{1-x}{z} = \bar{x}z + \tfrac{0}{0}x\bar{z}; \text{ or } = \bar{x} + \tfrac{0}{0}\bar{z};$$

which is the requisite result, viz. y in terms of x.

Most of the remaining generalizations which the Symbolic Logic introduces have been already fully expounded in the preceding chapters, but for the sake of aiding the total impression the more important of them may be briefly represented here.

The Schedule of Propositional Forms. The change here is more in respect of import and significance than in that of mere generalization, great as the latter is. Thus we entirely discard the common distinction between affirmative and negative, whilst we attach quite a new kind of importance to that between universal and particular. Thus for the old four propositional forms we substituted, as a beginning, the following eight[1]:

$$xy = 0, > 0.$$
$$x\bar{y} = 0, > 0.$$
$$\bar{x}y = 0, > 0.$$
$$\bar{x}\bar{y} = 0, > 0.$$

[1] It deserves to be pointed out that Lambert proposed a three-fold quantitative arrangement of propositions: "Auf diese Art dehnt sich

To these we might continue to add other forms, by combining two or more of the four kinds of class compartments. A slightly expanded table, containing sixteen forms, was given on page 190. How rapidly the number of possible forms mounts up when we thus group the possibilities furnished by the introduction of more than two class terms was indicated in Chapter XII. It was there seen that the mere declarations that such and such compartments were empty, when there were four class terms concerned, furnished no less than 4,294,967,295 cases. If we wished to retain some plan of grouping and arranging, which should be in accordance as far as possible with the old lines of arrangement, probably the best mode of doing so would be to proceed in the way proposed by Jevons and already referred to in the chapter in question[1].

Elimination. The process which we undertake under this name is a real generalization of familiar processes. The scope however of these processes, in the traditional treatment, is so slight that probably many logicians would have to pause and think, if asked whether, and under what conditions, rules could be laid down for the process of elimination of terms from a group of propositions. They do, of course, recognize such rules, but only to the trifling extent of eliminating one term out of three; the three being given two and two, in a pair of propositions of a specified kind, and the term to be eliminated being the one which occurs in both premises. What the Symbolic Logic proposes is the

die logische Arithmetic nur auf das *alle*, *etliche*, *kein* aus. *Alle* ist=1, *kein* is=0, *etliche* ist ein Bruch der zwischen 1 und 0 fällt, den man aber unbestimmt lässt" (*Architectonic*, I. 202). Though not worked out, this suggestion seems to me a very remarkable anticipation of some of those requirements of modern symbolists which even Boole had not duly realized.

[1] The reader may consult the discussion of this subject by Miss Ladd in the Johns Hopkins *Studies*.

problem:—Given any number of terms, contained in any number of propositions, assign a formula for the result of eliminating any number of those terms.

Reasonings. Here again we have a true process of generalization. A mode of stating the ordinary process would be the following: Given two propositions involving three terms, two and two, so that one term occurs in both; these propositions, remember, being selected from a strictly limited schedule containing four admissible types; find in what cases a third proposition distinct from those two, but drawn from the same schedule, can be inferred from them. The problem of the Symbolic Logic is: Given any number of non-quantitative propositions, of any type whatever, and involving any number of terms, assign a general formula indicating what class combinations can be established from them, what combinations can be negatived, and what combinations can receive no such solution either way. It will be seen therefore that whereas the problems of elimination and inference are almost exactly the same thing when regarded from the common or restricted point of view, they develop in very different directions when generalized.

CHAPTER XVIII.

CLASS SYMBOLS AS DENOTING PROPOSITIONS.

In a preceding chapter we had to say something about the existence of a certain 'universe of discourse', within which our symbols must always be considered to have their application, for the time being, confined. As to the *nature* of this universe we decided nothing: we gave it to be understood that this depended entirely upon the data, or rather upon the intention of those who employ the data. We merely insisted upon the condition that the various ultimate compartments or subdivisions were to be considered as being, within the limits assigned to the universe, mutually exclusive and collectively exhaustive. Our only assumption as to the mutual relations of x and $\bar{x}$ was, that, whatever x stood for,—and it might stand for almost anything we pleased,—$\bar{x}$ should stand for the rest which constituted the 'all' of which we proposed to take account.

Current logical prepossessions, fortified by the suggestions furnished in many of the examples which we have had occasion to employ, will have disposed most readers to put a special interpretation upon the nature of this logical universe, the insufficiency of which must now be pointed out. For one

thing it will probably be assumed that x and $\bar{x}$, &c. stand for classes of things as opposed to individuals; that is, that they are in their actual usage, what we have indeed commonly called them, *class terms.* And again, it will, I think, be tacitly assumed that the whole group of things referred to in the universe will generally be possessors in common of some recognized substratum of attributes, however they may be differentiated from each other by the presence and absence of other attributes.

The removal of the first of these restrictions will not give much trouble, since very familiar admissions on the part of the Common Logic have already prepared the way. It is fully recognized everywhere that the 'class' represented by a term in our propositions may be an individual: what is not so commonly recognized is the fact that the contradictory of that class may also be nothing more than another individual; that is, that we may reject the old doctrine as to the infinitude of a negative term. Of course, if our universe were very wide, the selection out of it of one individual would leave a miscellaneous host behind, but there is no necessity that the universe should be thus wide. Look, for instance, at the expression $x\bar{y} + \bar{x}y$. There is nothing to hinder us from restricting our universe here to Lord Salisbury and Lord Rosebery, and to the fact of their being in or out of office. If we mark the former of the two statesmen by x, the other will be marked by $\bar{x}$; and if we indicate the fact of being in office by y, then that of being out of office will be indicated by $\bar{y}$. Hence $1 = x\bar{y} + \bar{x}y$ asserts that the only things in our universe are Rosebery in office and Salisbury out of office. Or we might equally take x and y to stand for the two statesmen when in office, and $\bar{x}$ and $\bar{y}$ for the same men when out of office, and then interpret the same equation $1 = x\bar{y} + \bar{x}y$. When this statement is combined

with our standard formal condition, $1 = xy + x\bar{y} + \bar{x}y + \bar{x}\bar{y}$, we have $xy = 0$, $\bar{x}\bar{y} = 0$, indicating that both cannot be in office together, nor both out of office together. All this is simply a matter of application or interpretation of our rules, and, with one exception, in no way touches their formal validity. This exception will have to be discussed further on: it concerns the question whether we have any opening here for more than that 1 and 0 calculus which Boole worked out, and which we have seen will answer perfectly for universal propositions. That is, we shall have to enquire whether we can also introduce the notation $x > 0$, which we found to be necessary for the expression of particular propositions.

The other restriction which has to be laid aside may perhaps demand a little more explanation. When dealing with classes of things, especially in common conversation, it is so usual to group things together only when they are somewhat homogeneous, that we come to assume almost as a matter of course that the objects composing these classes have, so to say, a deep substratum in common, with only a superficial diversity. For instance, the kind of concrete class example which people would naturally fit into the formal framework $x\bar{y} + \bar{x}y$ would be, say, 'French who are not Catholics and Catholics who are not French'. Here it is clear that the great bulk of the attributes involved (i.e. those of humanity) are common to these two mutually exclusive classes, those which differentiate them being by comparison few. And this will mostly be the case[1], except when we designedly stretch our universe beyond the habitual reference of common thought, and make it approximate towards the logician's limits of what is conceivable.

[1] An amendment was recently moved in Parliament to the effect 'that any woman who, *if she were a man*, would be entitled to be placed on the register of voters, shall be so placed'.

The grounds of the belief that these represent the most appropriate kind of logical example lie far down in the necessities of language, and in the physical and mental causes by which these have been produced. They seem to me connected with the general practice of making the subject and the predicate in most cases roughly correspond respectively to substance and attribute. In affirmation at any rate, if not in negation, we commonly regard the subject as a *thing*, or class of things, endowed with miscellaneous and indefinitely numerous attributes, the function of the predicate being to modify the subject by adding on one or more fresh attributes. The leading conception is throughout that of a substratum of permanent qualities with a relatively small number of differentiating ones. The ordinary logician has done something to liberalize our usage in this respect, but not with very great success, the grammarian being too powerful for him. We encounter a difficulty, for instance, in the process of conversion, for when we want to make the subject and predicate change places the proprieties of language greatly thwart our logical rules. 'Some men are passionate', has to be changed into 'Some passionate (persons, things, or what not) are men'. We feel it necessary, so to say, to *weight* the subject, since it sounds awkward to have a slight subject with a heavy predicate attached to it[1]. Propriety of language suggests that the subject should contain the bulk of the attributes; should, in fact, be by comparison a substance of some kind modified by attributes.

Recur for a minute to the example about the French Catholics. This might be written in the form: 'Men (whether French and not-Catholic, or Catholic and not-French)': in

[1] De Morgan has humorously enquired why we should not speak of "the horseness of the speed" as well as of "the speed of the horse." (*Camb. Phil. Trans.* x. 340.)

which the bulk of the attributes, namely those conveyed by 'men', are common to both the distinct classes. So customary is it thus to group homogeneous things, that the forms of language tend to keep up the illusion of it even when there is no reality underlying them. They succeed in doing this in one or two different ways. Let, for instance, our terms and their contradictories stand for the presence and the absence of two men A and B; what constant common element is there throughout the expression $x\bar{y}+\bar{x}y$? Its two elements stand respectively for 'the presence of A and the absence of B', and 'the absence of A and the presence of B'; and these have really nothing in common. But when we group these together we should very commonly talk of 'the case of' A being present and B absent, just as we talked about 'the men' whether Catholics or not. But the one phrase does, and the other does not, imply a real substratum of attributes, that is, of relevant attributes, in common.

In other instances this device of language is less thinly veiled. Thus if $(zx\bar{y}+\bar{z}\bar{x}y)$ stand for 'cold and wind without rain, or not-cold and rain without wind', we should very probably apply the common word 'weather' to both cases, by speaking of the weather being cold and windy and not rainy, or not cold and rainy and not windy. But the cold and wind and the rain together *are* the weather, or at any rate very nearly comprise all that is signified by the word. The constituent elements therefore in the logical expression have really little or nothing in common, and the common word prefixed to them all alike is scarcely more than a device of language.

Whether or not this special interpretation be accepted, the above examples will equally serve to call attention to an extremely important generalization upon which we now have to insist. It is indeed difficult to exaggerate the importance

of the step before us, since its due realization is, to my thinking, one of the most valuable speculative advantages to be gained by the study of a generalized Logic. It has come under our notice incidentally more than once already, but must now be deliberately insisted on.

Our present position then is this. Starting with the familiar conception of classes composed of an indefinite number of individuals, we found that we could conveniently express the desired logical relations of these classes by aid of a set of symbols, borrowed, as it happens, from mathematics. But those who call in the aid of symbols generally find that they have got possession of a machine which is capable of doing a great deal more work, and even work of very different kinds from what they originally expected from it. To the mathematician, of course, this is perfectly familiar. Indeed the history of the formal part of his science is in great part an account of the successive extensions opened out by new interpretations and applications of the same set of formal rules[1]. Sometimes the machine needs a slight structural alteration to enable it to perform its new work:—thus, in mathematics, in order to make our algebraic symbols do Quaternion work we have to suspend the commutative law. But, as a rule, we retain the old laws unchanged, merely looking out for new work which they are fitted to perform.

This then is the step now before us in Logic. Look again at the fundamental formula, $1 = xy + x\bar{y} + \bar{x}y + \bar{x}\bar{y}$. It may be quite true that when we first appealed to it we only had in view such classes of things as those denoted by general names. But a little reflection soon convinces us that the symbols have wider capabilities. All that they imperatively

[1] In accordance with what is sometimes called the Principle of the Permanence of Equivalent Forms.

demand is that the various elements $x\bar{y}$, $\bar{x}y$, and so forth, shall be mutually exclusive and collectively exhaustive, and that they shall obey the few simple laws of operation described in the second chapter. Our formal framework may have been originally devised to cover a system of classes each composed of a multitude of individuals, but now that we have acquired it we find that it is best regarded as simply a scheme of contradictories of a certain kind, subject to certain laws, and that it will therefore cover any set of contradictories which fulfil the same requirements.

As we do not at present propose to admit anything that can be called a structural change in our symbolic machine, by modifying the laws of operation, all that we have to do is to see how many of the tacit or customary restrictions upon the nature of a 'class' can be laid aside without affecting the validity of our symbolic operations. Now a moment's consideration will show that such restrictions as those noticed towards the commencement of this chapter are of no significance whatever. Whether $x\bar{y}$ and $\bar{x}y$ have any essential, or even important attributes in common; whether they stand for indefinitely numerous groups, for limited groups, or for individuals; is indifferent. Accordingly we may generalize without hesitation up to this extent.

But a good deal more than this can be done by our discarding any obligation on the part of the symbols directly to represent material objects at all. Let x, for instance, denote the truth of a given proposition and $\bar{x}$ its falsity. This assumption (as will presently be shown in more detail) will fit in excellently with our requirements. We should then have to interpret the expression $x\bar{y} + \bar{x}y$ as representing the alternative of the truth of one, and one only, of the two propositions referred to by x and y. Again, we might understand by x something abstract, but more complex, by re-

garding it as representing the validity of some whole train of argument. In this case $\bar{x}$ stands for the invalidity of the same argument, that is, for its failure to establish its conclusion, whether that conclusion in itself be true or false. Combine these symbols with others of a like significance, and we have at once expressions for the combination of arguments sound and unsound. Thus $x\bar{y}z$ might stand for the validity of the two arguments indicated by x and z, and the invalidity of that indicated by y.

Again, we may set the symbols to stand for the trustworthiness of assigned witnesses. Thus if x stand for some man's trustworthiness, $\bar{x}$ will stand for the contradiction of this, viz. for his untrustworthiness, and therefore indicates that we have so far no grounds of opinion either for or against the fact testified. A distinct case from this last is produced by letting x stand for the *truthfulness* of the witness, that is, for the fact that his statement is actually true. This leads to some difference of interpretation throughout our use of the pair of contradictories x and $\bar{x}$. The antithesis now is not merely between the fact of a thing being proved and not being proved, but between its having actually happened and not having happened. Hence combinations admissible under the former interpretation may become inadmissible under the latter. To say that the witness is untrue implies that his statement is false; to say that he is untrustworthy leaves the way open to either the truth or falsehood of his statement. Problems belonging to these latter classes play a large part, under numerical treatment, in the Theory of Probability[1],

[1] A very good account of the meaning and consequences of the particular distinction in question is to be found in De Morgan's *Formal Logic* (p. 191), and also in Boole's *Laws of Thought*. It was in fact largely for the purpose of improving the Calculus of Probabilities that Boole devised his system.

but do not deserve any special notice in a purely logical treatise.

It must not be assumed that the interpretations above indicated represent the only kinds available for us, or even the only fully suitable kinds. It would be rash to maintain that a symbolic apparatus will only do those kinds of work for which we may have happened hitherto to find it capable. But the reader will readily trace the same formal antithesis between 'is' and 'is not', and the permanency of the same laws of operation, throughout the whole field of available application. It must be clearly understood that none of these various interpretations can be regarded as giving rise to anything deserving the name of a distinct scheme or system. Symbolically, the scheme is absolutely one and the same throughout:—with one exception noticed further on in this chapter. Indeed, if we abstract sufficiently, we can also say that there is only one and the same *interpretation* throughout, for in every case we deal with the various combinations in which affirmations and denials may be arranged. To the one formal framework built up out of x and $\bar{x}$ and the like, corresponds the verbal framework which may be built up out of *Yes* and *No* variously combined and applied. Symbolically and verbally, alike, there is in each case a primary and necessary scheme of possibilities, which is limited by the experimental and material conditions afforded by the data.

Amongst all the various interpretations which are thus opened out to us in the use of our symbols, there is one which deserves particular notice, both from the generality of its possible application and the special nomenclature to which it leads. It is that in which the symbols stand for propositions, i.e. not for the propositions themselves, but for their truth or falsehood. That some such interpretation could be

imposed upon the symbolic processes did not escape the penetration of Boole[1]. He devotes a large part of his volume to their consideration, under the head of 'secondary propositions', and discusses the definitions and deduces the laws of operation independently for them, as he did at the outset for the more familiar application to classes of things. This mode of treatment, combined with his rather far-fetched and unnecessary doctrine (as it seems to me) that propositions of this secondary kind were peculiarly interpretable in terms of *time*, has probably contributed to make some writers forget how completely Boole had grasped the possibility of this interpretation of his symbolic methods[2]. There is much in

[1] In a certain sense the employment of single letters to denote, not single terms only, but rather whole propositions, has often been adopted in Logic. Some writers have carried this out very systematically, with a special notation to denote the contradictory of any proposition. For instance, Maass (*Grundriss der Logik*, 1793) has used *Arabic* letters to stand for ordinary subjects and predicates, and Greek letters to stand for whole judgments or propositions. Thus, if we wished to represent the whole judgment 'all a is b' as a single element, we should put the letter α, say, to stand for it, and $n\alpha$ to stand for its contradictory. Some of the deductions on this mode of notation will be familiar to readers of De Morgan and other modern symbolists: e.g. 'when β follows from α, and is false, then α must be false too; when β follows from α, γ, δ... together, and is false, then α, γ, δ... cannot all be true'; and so on (p. 126).

[2] On this particular point the case seems to me to be as follows. Boole gave two distinct interpretations to his symbols: in one of these x is regarded as a class of things, and in the other as a proposition. Those who have proposed modified schemes have, broadly speaking, confined themselves to one only of these interpretations (or rather to a part only of one, for they have not admitted any special signs for *inverse* operations). Thus Mr McColl makes it a "cardinal point" of distinction in his scheme that "every single letter, as well as every combination of letters, always denotes a *statement*", and this he conceives to "necessitate an essentially different treatment of the whole subject". Jevons, on the other hand, though not expressly defining his position here, invariably confines his interpretation, unless I am mistaken, to what may be called the material class view of the symbols. Prof. Schröder, whilst regarding the

his treatment of this part of the subject which invites criticism, but as we are here concerned with Symbolic Logic itself rather than with the particular opinions held by its main originator, I pass these by wherever it is possible.

It may be well to go a little more into the details of this interpretation, starting with the most general symbolic statement where two terms are combined, viz. $xy+x\bar{y}+\bar{x}y+\bar{x}\bar{y}=1$. As there are supposed to be two propositions involved, we may at once, for the sake of distinction and brevity, call them the propositions x and y. We might have described the propositions as A and B, or as X and Y, respectively, so as to employ different symbols for the proposition itself and for the intimation of its being true; but the two-fold application of the same symbol need not lead to confusion in the processes of working. We say then that the symbol x standing for the truth of the one proposition, and y standing for that of the other, the left side of the symbolic equation represents the only four possibilities. Its four component elements stand for the truth of both propositions (xy), the truth of x and falsity of y ($x\bar{y}$), the truth of y and falsity of x ($\bar{x}y$), and the falsity of both ($\bar{x}\bar{y}$).

What then is here the signification of *unity*, or 1? In the general formulæ as hitherto interpreted, unity stands for the sum-total of all the individuals composing all the possible classes; or, to speak more accurately,—since we avoid enumerating individuals,—it stands for the sum-total of all the possible class combinations. The best verbal account per-

class interpretation as the primary one, has also given a thorough investigation of the propositional interpretation.

My only complaint against Boole on this point is that even he did not generalize sufficiently. Doubtless he directed attention to what are far the most important interpretations, but I prefer to regard even these as constituting only a portion of what is potentially an indefinitely wide field of application.

haps, in the present application, is given by saying that it stands for the sum-total of possibilities or possible cases. The formula asserts that the four cases above enumerated, as to the truth and falsity of x and y, exhaust the whole field of possibilities. Of course the actual limits of this field, that is, of this universe, have to be assigned on the same principles as in the more familiar case. It may be that the propositions themselves are necessarily true or false: in which case the universe, so far as any of its time conditions are concerned, is altogether unlimited. It may be that they are true only under certain conditions,—conditions of time, of place, of circumstance, or what not: the universe of possibilities is then limited by these considerations. But it does not seem to me to be correct to say that in the one case *unity* stands for eternity, and in the other case for a definite portion of time; that is, to interpret as if time itself, in whole or in part, could constitute our universe. I regard the universe as made up of the sum of possibilities introduced into the question, whether these comprise combinations of the truth and falsity of what are called eternal, or only temporary propositions. The determination of the temporal range of our propositions must really rank as a part of our data: it is one of the elements of information which we must take for granted in the solution of the problem: our general symbolic laws can no more throw light upon the limits within which they are to be held to apply, than they can upon the special signification which it is there proposed to assign to our symbols.

Suppose, for instance, that x stands for the truth of, 'There is a God', and y for that of, 'The just will be happy hereafter'; then $xy+\bar{x}\bar{y}=1$, will stand for the statement that the only admissible possibilities are the truth of both these assertions and their falsity. We are excluding the

possibility that there can be a God and the just not be happy, and that the just should be happy and yet there be no God. But we might equally put x for the truth of the proposition, 'It will rain in Cambridge to-morrow afternoon', and y for that of 'I shall stop at home and write'. In each case alike the significance of this symbol 1 seems to me to be the same. It exhausts the possibilities which the forms of thought and the matter of experience together allow under the circumstances.

So with the sign for *nothing*, or 0. Just as it is elsewhere interpreted 'no individual' (of a class), so here it signifies 'no possibility'. At least this seems the best verbal rendering of the case; though, as the symbolist has to reckon with the grammarian, he is often somewhat put to it when trying to express himself unambiguously. We may say then that the expression, $x = 0$, implies that within our determinate field, that is, subject to our special conditions, there can be no admission of the truth of the proposition x.

As we have said, x stands for the truth of a proposition: to put $x = 0$ is a way of saying that there is no truth in it, that the truth of it is then and there excluded[1]. If the

[1] Dr Husserl gives as the signification of 1, on the interpretation in question, "the validity of the proposition that of two contradictory judgments one must be true and the other false." I apprehend that one object of this (as it seems to me) rather far-fetched interpretation is to meet the case of a formula which occurs in the subsumption or implication rendering of our system, viz. that, whatever x may be, $x \nless 1$. "The truth of every judgment presupposes that of the above principle: without it there would be no distinction between truth and error. If then any judgment is true, the above principle holds good." The symbol 0 then stands for the impossible case of the invalidity of the above principle.

I do not see any necessity for such an interpretation, and it appears to me that we thus exclude the much more necessary applications of these symbols given above. We could not

reader feels any difficulty in thus employing a symbol to stand for the truth of a proposition, without at the same time committing ourselves to the fact that the proposition is true, he must remember that this is simply another version of precisely the same general distinction that we had to face at the outset. It is in fact, the old distinction between representing terms and representing propositions. What we have said hitherto was, Let x stand for the class denoted by a given term: but this in no way committed us to the conclusion that any such class was to be found. The existence, or otherwise, of the class only comes up for decision when we introduce a proposition. If we put $x=0$, we deny its existence: if we put $x>0$ we assert its existence.

What then is meant by the equation $x=1$? Simply that the proposition is true. As we have said, the term x by itself stands for the truth of the proposition, and the change which the equation, as contrasted with the term, effects is to convert this problematical truth into actual assertion of truth. The interpretation here is closely parallel with that which was demanded when we were dealing with classes. There, as we saw, x by itself stands for the class:— whether existent or not. Put $x=1$ (as distinguished from $x>0$), and we rule at once that the class is established exclusively: i.e. that $\bar{x}$ is abolished, and that therefore whatever other classes there may be, such as y, z, &c., these must all, if existent, coexist with x. So here: x standing for the truth of the proposition, by $x=1$ (we cannot now, as will presently be seen, put $x>0$) we mean that this truth holds the field, so to say. The falsity of the proposition is

then equate the four possibilities to the total represented by this symbol 1: we could not intelligibly put $x=1$ or $x=0$. (See Husserl, *Folgerungs-calcul.*)

absolutely excluded, and therefore, in correspondence with the last case, whatever other propositions may be introduced, such as y, z, &c., their truth or falsity must necessarily coexist with the truth of x. In other words, so far as the terms x and y are concerned, we regard the truth and falsity of the propositions as being of the nature of contradictory attributes. As soon as x has been ruled $=1$, then we are left with the two possibilities $y=0$, $y=1$: whichever of these may prove to be correct it must certainly be held with x.

The reader will bear in mind that all this is not a process of original legislation, but simply one of interpretation. We are not recurring to the business of deciding the general signification of our terms and operations:—this was done once for all at the outset:—but merely considering now what is the most suitable phraseology for translating and describing our results in a certain well-marked and distinctive group of cases. We see at once, for instance, that $x+\bar{x}=1$ should be interpreted as saying that either a proposition or its contradictory must certainly be true, and that $x\bar{x}=0$ asserts that a proposition and its contradictory cannot both be true. Again $xy=1$ (implying, as it does, $x=1$, $y=1$) necessarily leads to the truth,—i.e. the exclusive truth, in the sense already explained,—of both propositions. But $xy=0$ only insures the denial of the particular compound xy; that is, it tells us directly that the propositions are inconsistent, and therefore indirectly that both propositions cannot be true, so that it leaves three possibilities open: viz. that either of the two, or both together, may be false. We see this more plainly by looking at the alternative or disjunctive side. From $1=xy+x\bar{y}+\bar{x}y+\bar{x}\bar{y}$ blot out xy, and we have as remainder $1=x\bar{y}+\bar{x}y+\bar{x}\bar{y}$. That is, if we deny that both the propositions are true, we are reduced to

the three alternatives, that one at least, or both, must be false. Blot out xy and $x\bar{y}$ and we have $1=\bar{x}y+\bar{x}\bar{y}$. That is, if we can deny that 'x and y are both true', and also that 'x is true and y false', we are led to the conclusion that either x is false and y true, or else that x is false and y false. In other words x must be false, and y doubtful, so far as this information goes.

The case of the combination of two propositions presents no real difficulty, though it needs some explanation in certain extreme cases. We saw, in an early chapter, that $x\bar{y}=0$ (or its equivalents, $x=xy$, $x=\frac{0}{0}y$) might stand for the proposition, 'All x is y', or, if we preferred so to phrase it, for 'If anything is an x it is a y'. Other admissible renderings were also pointed out. When x and y stand for propositions the same equivalent forms of course demand explanation. The most natural rendering for the negative form is the simple denial that x can be true and y false. That which most readily fits the affirmative forms is the ordinary conditional or hypothetical, 'If x is true then y is true'.

In offering these renderings, however, we must remember that there is a certain extreme case in which we have to meet a good deal of adverse association. The two interpretations meet the shock at somewhat different points: the former when it has to admit that 'All x is y' covers the case of there being no x or no y, and the latter in sanctioning that 'If x is true y is true' covers the case of y having no connexion whatever with x provided only y is known to be true. That is, the former has a difficulty in carrying the extreme values $x=0$, $y=0$, and the latter in carrying the extreme value $y=1$. The general propriety of admitting, or rather the necessity of insisting upon the admission of, both these extreme values, has been fully discussed already (see chh. VI., VIII.) and needs no further justification. But

the language in which the result is sometimes expressed, upon the special propositional interpretation now under review, needs notice, since it involves a certain violence to popular associations. We have seen already that we must stretch the language of hypothesis in the phrase, 'If x then y', to reach the case of y being certain: we must now claim to do the same with the word *implication*. We must admit that the phrase 'x implies y' does not imply that the facts concerned are known to be connected, or that the one proposition is formally inferrible from the other.

This particular aspect of the question will very likely be familiar to some of my readers from a problem recently circulated, for comparison of opinions, amongst logicians. As the proposer is, to the general reader, better known in a very different branch of literature, I will call it *Alice's* Problem. It may be stated successively in the two following forms:—There are two propositions, A and B. Let it be granted that (i) If A is true, B is true. Also let there be another proposition C, such that (ii) If C is true then if A is true B is not true. There was supposed to be a dispute as to whether, under these circumstances, C could be true: one side maintaining the affirmative and the other the negative. The second form is this;—There are three men in a house, Allen, Brown, and Carr, who may go in and out, provided that (i) Allen never goes out without Brown, and that (ii) If Carr is out, then, if Allen is out Brown is in. Under these circumstances it was similarly disputed whether Carr can ever go out.

By any reader who has followed the general reasoning of this chapter it will be readily admitted that these two problems are merely different verbal renderings of one and the same symbolic process. Nor will those who accept the main principles of the Symbolic Logic feel the slightest

difficulty in interpreting the result in the extreme case in which any one of the terms, A, B, C, is $=0$.

The problem symbolically is simply this: We have three terms, A, B, C, and are told that (i) $A\bar{B}=0$; (ii) $CAB=0$. We are then asked, Must $C=0$? It is almost intuitively obvious that, on all the principles hitherto laid down, the answer must be in the negative. There were originally eight possibilities. Of these, two are ruled out by the first premise, viz. $A\bar{B}C$, $A\bar{B}\bar{C}$; and one by the second, viz. ABC. This leaves two cases of C surviving, viz. $\bar{A}BC$ and $\bar{A}\bar{B}C$, or, more briefly, $\bar{A}C$. As regards the interpretation of this result, it depends upon the special signification put upon the symbols. If these denote the truth and falsity of propositions, we interpret by saying that C can be true when A is false, whether or not B be true. If they denote the absence and presence of individuals, then we interpret by saying that C may go out when A remains at home, whether or not B goes out. But any other of the significations mentioned on page 433 as yielding a true dichotomy, of the Yes and No type, would be equally available. And in all cases alike we must be prepared to accept and interpret the results when one or more of our symbols are equated to zero.

The difficulty, of course, to the ordinary logician is that under a certain condition (that of C) we combine the two propositions 'If A then B' and 'If A then not B'; and this is to him a serious breach of convention, as it implies that the subject of a proposition may be non-existent. But to us there is no difficulty in such an implication, whether under such a condition as C, or without any condition.

It was intimated at the outset of this chapter that every statement could be thrown into the form of the truth of some corresponding proposition. Whether it will be convenient or

not to adopt this plan depends upon circumstances. Sometimes it seems to be a matter of pure indifference, amounting at most to a trifling verbal change; sometimes such a rendering represents a distinct economy of language and labour; sometimes the reverse. As an instance of the class of cases in which either mode of interpretation does equally well, we might take the following:—

It will either blow a gale to-morrow or the mail will start.
He will go if the mail starts.
∴ If he does not go it will blow a gale.

This may be written out symbolically, thus,

$$\begin{cases} x + \bar{x}y = 1, \\ \quad\; y\bar{z} = 0, \end{cases}$$

$$\therefore \bar{x}\bar{z} = 0 \text{ or } \bar{z} = \tfrac{0}{0}\,x.$$

(This may be worked out as follows:—substitute the value of y, viz. $y = \frac{0}{0}z$, obtained by the second equation, in the first, and we have $x + \frac{0}{0}\bar{x}z = 1$, or $\bar{x} = \frac{0}{0}\bar{x}z$, $\therefore \bar{x}\bar{z} = 0$.)

Here it surely represents such a trifling difference as hardly to be worth notice in a system of Logic, whether we say, Let x represent the blowing of a gale, and y the starting of the mail; or say, Let x represent the proposition that it will blow a gale and y the corresponding proposition about the starting of the mail.

Now consider a case in which the proposition explanation seems decidedly less appropriate. Take the familiar syllogism;—

No *B* is *C*,
All *A* is *B*,
∴ No *A* is *C*.

How would any one proceed here who was merely told that the proposition, or the truth of a proposition, was to be

the ultimate element denoted by his terms? He would probably say, Let x, y, z, stand for the truth of these three propositions respectively; at least he might start in this way. But at this point, as is readily seen, we are brought to a standstill symbolically; for nothing having been assigned by way of connection between x, y, z, no conclusion about one of them can be elicited from the other two. All we can do is to *indicate* the syllogistic process, not to analyse it. We can only express the fact that the truth of both x and y is inconsistent with the falsity of z, by writing $xy\bar{z}=0$. If from $\bar{x}=0$ and $\bar{y}=0$ we could infer directly anything about z, we should have produced a great simplification; but clearly the only way of deducing this is to go through the syllogistic process. If we did this we should be found to be taking a rather roundabout course by introducing any special symbols for the entire propositions. The simplest plan would of course be to write our propositions down fully in the ordinary way, putting x, y, and z, for A, B, C themselves directly instead of for propositions about them, thus;—

$$\begin{cases} yz = 0, \\ x\bar{y} = 0, \quad \therefore xz = \tfrac{0}{0}\, yz = 0. \end{cases}$$

For these represent the real relations between the propositions, and they must therefore be introduced somehow if any conclusion has to be drawn.

Accordingly those who adhere to the uniform propositional rendering of our symbols have to state the syllogism somewhat as follows[1];—Let x stand for the statement, made in reference to any object, 'it is A', y for 'it is B', z for 'it is C'. The premise 'No B is C' then takes the form $yz=0$ or one of its equivalents; 'All A is B' becomes $x\bar{y}=0$, and so on.

[1] See, for instance, Mr McColl's rendering in *Mind* (No. XVII.). Of course the above notation is not his.

In fact this rendering simply gives us the ordinary symbolic statement over again, with a slightly different and, as it seems to me, more cumbrous interpretation attached to it.

As another illustration of the comparative indifferency, in most cases, of these two interpretations, take the following example. "Given that (1) whenever the statements a, b, x are either all three true, or all three false, then the statement c is false and y is true, or else c is true and y is false; (2) that whenever d, e, y, are either all three true or all three false, then the statement a is false and x is true, or a is true and x is false. When can we infer, from these premises, that either x or y is true?" (McColl, in *Educational Times*, No. 6,206:—the problem has been already discussed, as a merely symbolic process, in the chapter on Elimination: see page 295).

As a, b, c, d, &c. are only taken account of here under the two-fold aspect of *true* or *false*, it is plain that all the resultant combinations will precisely correspond to those which would arise had we regarded the symbols as denoting qualities which could be *present* and *absent*. But had the question been thus phrased, we should naturally have said that the presence and absence of a quality meant that the object did or did not belong to the corresponding class, and adopt at once the class-interpretation.

We may therefore write down the premises as follows; in the form of negations;—

$$\begin{cases}(abx + \bar{a}\bar{b}\bar{x})(cy + \bar{c}\bar{y}) = 0,\\(dey + \bar{d}\bar{e}\bar{y})(ax + \bar{a}\bar{x}) = 0.\end{cases}$$

Multiply out, and arrange in the form

$$Axy + Bx\bar{y} + C\bar{x}y + D\bar{x}\bar{y} = 0,$$

and we find,

$$A = abc + ade,$$

$$B = ab\bar{c} + ad\bar{e},$$

$$C = \bar{a}\bar{b}c + \bar{a}\bar{d}e,$$

$$D = \bar{a}\bar{b}\bar{c} + \bar{a}\bar{d}\bar{e}.$$

Now we have already seen (v. page 316) that when the question here simply is, Under what circumstances do x or y occur, the answer is given by $D\bar{x}\bar{y} = 0$: i.e. Wherever there is D there is x or y. That is, wherever we have $\bar{a}\bar{b}\bar{c}$ or $\bar{a}\bar{d}\bar{e}$ we have x or y. Interpreted into the language appropriate to the propositional explanation we have; If either a, b, c or a, d, e are all false then either x or y is true.

It is equally obvious that if the question had been, When will either x or y be *false*? we should have selected A, and said; Whenever either a, b, c, or a, d, e, are all true, x or y must be false.

It may be pointed out here, in illustration of the remarks on Elimination of two symbols simultaneously (see page 390), that it is clear at a glance that $AD = 0$, so that x and y cannot be both eliminated to any significant purpose.

So far, in this chapter, we have discussed the question as a '1 and 0 problem': that is, the symbolic framework applicable to universal propositions has been found to apply here, but we have had no occasion to introduce the notation belonging to the particular proposition. This point must now be enquired into.

The reader's attention must first be recalled to the fact that, in order to preserve the distinction between terms and propositions, we do not say that x stands for 'the proposition is true' but for 'the truth of the proposition'. The truth is regarded as a sort of attribute which may be present or not:

that is, xy represents 'the truth of both x and y', which is contemplated as one of the four possible alternatives whose sum-total $= 1$. It stands for a truth, so to say, which is not yet established; just as on the class-interpretation it stands for a class whose existence is not yet established. For such establishment we must in each case wait till a mere term is transformed into an equation.

It seems quite clear, then, that so long as we regard the truth in question as unconditioned by time or place we can find an opening for 1 and 0 only, as on Boole's own scheme. There can be nothing intermediate between simply negating xy, and positing it to the exclusion of the other three terms. On the *class* interpretation, when we put $x + \bar{x} = 1$, we mean that the total may be made up by contributions from both x and $\bar{x}$, for we admit the case $x > 0 \therefore \bar{x} < 1$. But on the present interpretation to admit x at all, is to reject $\bar{x}$ absolutely. If therefore we accept xy as true, we *ipso facto* rule out $x\bar{y}$, $\bar{x}y$, $\bar{x}\bar{y}$. So, when we rule out xy by equating it to zero, we have a strict disjunction before us: we know that one, and one only, of the remaining three can be established, but we have no means of saying, so far, which it will be. Suppose, for instance, x stands for the truth of 'it rains', y for that of 'hay is being made', at some place and time: then it is clear that if we put $xy = 0$, we merely deny that both statements are true, and are left in face of three alternatives of which one must be true.

The distinction may be illustrated by appeal to a diagram. Draw the two circles, and scratch out xy. Then on the common interpretation we admit that any two, or all three, of the remaining compartments may possibly be occupied. But suppose that we had to locate *a single point* somewhere on the saved space: it is clear that to place it in $x\bar{y}$ is equivalent to denying that it can be in $\bar{x}y$ or $\bar{x}\bar{y}$. This

corresponds to the solitary admissibility of a single alternative in the complete scheme of possibilities.

It may be conceived, however, that, by the admission of time and place conditions into our propositions, this restriction can be obviated. Let us say, for instance, that x shall denote the truth of the statement, 'It rains somewhere in Cambridgeshire'; or 'It rains sometime in the month', or 'It rains somewhere, sometime, in the county and during the month'. Now, though it is quite certain that, as regards the facts themselves, the occurrence of rain at some time or place is clearly compatible with its presence or absence at other times or places, yet the truth of the respective propositions gives no opening to any corresponding alternatives. If, for instance, x and y stand for the truth of the propositions, It rains in Cambridgeshire, Hay is made somewhere there; then $\bar{x}$ and $\bar{y}$ stand for the falsity of these statements. Thus put $xy = 0$. This means that throughout the county, we shall not find both rain and haymaking:—e.g. it cannot have rained at Ely and hay been made at Wisbech.

Now at first sight it seems as if there was an opening for the form $xy > 0$, as distinguished from $xy = 1$, and that this might be interpreted as, Rain occurs and also haymaking: i.e. one event in one place and the other in some other place. But this, on closer inspection, fails to establish the requisite distinction. What $xy > 0$ asserts is that the truth of both x and y is secured, therefore the falsity of each is absolutely excluded. That is, to put $xy > 0$ is here exactly the same as putting it $= 1$: we cannot save it without destroying the other three alternatives. No modification, by appeal to time or space, in the *import* of our proposition can extract more than the two alternatives of its truth and its falsity.

It will be remembered that in a former chapter we said

that under certain circumstances 'and' and 'or', as connecting links in the complex subject of a proposition, might be equivalent. The reader will now see that this may be the case on the propositional interpretation. The expression $x + y = v \, . \, z$ refers the aggregate class $x + y$ to the class z. If x and y are true classes, comprising an indefinite plurality of objects, we naturally phrase this as '*x and y* are *z*'. But now suppose that we have only an individual (or collective whole) to locate in z, then this would have to be 'either an x, or a y, or both x and y'. Add on the further assumption that no x can be y, and we are brought to '*x or y* is *z*'. That is, the above suppositions have reduced what must be called the general case of an aggregate plurality into the special case of a disjunctive individuality. The requisite conditions for this are; (1) the universal postulate of Symbolic Logic that any term may $= 0$; together with the special conditions, (2) that the occupant of the class must be an individual, (3) that such individual cannot be both x and y. The second of these assumptions is practically made when we make x stand for the truth of a proposition; the third is a special and arbitrary restriction introduced by Boole. Under these three conditions it will be seen that in the expression, 'Everything in class x and everything in class y is a z', we may substitute 'or' for 'and'. That is, whereas the general case would be naturally stated with an 'and', this special modification of it would be naturally stated with an 'or'.

In this chapter I have given my own version of the propositional interpretation, which does not altogether coincide with that of Boole. A few words, however, about this latter may here be added. The following extracts will serve to show the peculiar sense in which he regarded the universe as being here made up of portions of *time*. "'If the proposition X is true, the proposition Y is true'. An undoubted

meaning of this proposition is that the *time* in which the proposition X is true is *time* in which the proposition Y is true" (p. 163). "As 1 denotes the whole duration of time and x denotes that portion of it for which the proposition X is true, $\bar{x}$ will denote that portion of time for which the proposition X is false" (p. 168). I find extreme difficulty in carrying out this interpretation intelligibly. Consider first the case of a proposition in the present tense, referring to some event or occurrence which continues to happen during a certain portion of time. The most natural meaning, I think, would be to suppose some one incessantly repeating a proposition in the present tense, e.g. "it rains", and to recognize this statement as true whenever rain was actually falling, and as false whenever it was not. But then what need is there here to introduce the machinery of a proposition at all? Our former examples about the rain and hay show that we can better let the symbols stand for the events themselves.

Take next the case of some event which is supposed to have happened in the past, or is expected to happen in the future, the corresponding proposition being in the appropriate tense. As a concrete example, suitable to illustrate the first of the above extracts from Boole, we might say, Let X stand for 'Francis wrote the Junius letters' and Y for 'Francis will be proved to be the author'. It is clear that the present-tense explanation will not answer in any such case as this: the two events are of brief duration, and widely separated in time. If we are to take as our test the time during which any one could have accepted the propositions in the tense in which they stand, then X is true (if true at all) since 1769; and Y, from that date up to some unknown date in the future. And if, finally, we take a more objective test by supposing some one with perfect prospective and

retrospective powers, then both statements, if true at all, must be considered true for all time; in the sense that such a person would always admit their truth.

If we take the first interpretation, which seems on the whole the most literal one, we can of course readily apply it to events which happen continuously through certain portions of time. We consider the proposition as 'true' during the time only when the events to which it refers are happening, and 'false' at all other times. For instance, if X represents 'it rains', and Y 'the sun shines', at some assigned place; then, as we have already seen, we can make such an assertion as $1 = X\bar{Y} + \bar{X}Y + \bar{X}\bar{Y}$, i.e. $XY = 0$. But then we can do just the same in respect of place; the proposition being considered true *where* it rains, and false where it does not. But I find it very difficult to assign a reasonable sense to either interpretation when the event referred to, instead of being continuous through some portion of time or over some portion of space, is definite and isolated. If the event really happened as stated, then it seems to me that we ought to regard the proposition as being always and every where 'true':—grammatical difficulties of tense being laid aside.

CHAPTER XIX.

INTENSIVE INTERPRETATION.

THROUGHOUT this volume we have adhered as much as possible,—allowing for the generalizations of the last chapter,—to an exclusive class interpretation of our symbols. With us, x, y, z, stand for classes, not for attributes: they denote the extension, not the intension of our terms. But those who have read the notes on the preceding pages will have perceived that this view of the subject has been by no means universal[1]. From the first commencement of a symbolic treatment of Logic there has been a disposition,—preponderant at first, but gradually growing less frequent,—to accept an intensive interpretation of the symbols em-

[1] I think it may be said that the true intensive view is practically abandoned now, though verbally it is from time to time espoused. Thus Jevons declared (*Pure Logic*, p. 2) "I do not wish to express any opinion here as to the nature of a system of logic in extent":—but I feel little doubt that his readers will agree with me that his practical treatment is throughout one of extent. Prof. Schröder expresses the current symbolic view when he admits, "In diesem gänzlichen Absehen vom Inhalte [intent] der Begriffe liegt nun allerdings eine Einseitigkeit" (Review of Frege in the *Zeits. für Math. u. Phys.* xxv. 3).

ployed. This attitude was generally adopted tacitly, and often almost unconsciously, in adherence perhaps to traditions of the common logic: but with the single exception, I think, to be noticed presently, the consequences of such an explanation were never fully worked out. As it is essential to understand something of the contrast involved between these two distinct views, if only for interpreting the processes and formulæ of some of the earlier writers, no excuse is required for going into somewhat of detail here.

The first difficulty which meets us in such a discussion concerns the signification to be assigned to the intent or connotation of a term. In a general way, of course, all writers are agreed that *intent* looks to the number and nature of the attributes involved in a term, just as *extent* does to those of the individual objects referred to by the term. But as to how many attributes are to be included as comprising this intent, there is the widest difference of opinion. Some (e.g. Bain) incline to widen the number so as to make it comprise all the permanent attributes known to be possessed by an object, even the most recondite of those which Science may have recently discovered. Some go yet further. Thus Jevons asserts that "a term taken in intent has for its meaning the whole infinite series of qualities and circumstances which a thing possesses...some may be known...the infinite remainder are unknown" (*Pure Logic*, p. 4). Others again, on the other hand, are disposed to curtail the number to the minimum required to distinguish the object from others. An intermediate view, such as was held by Mill, is probably the most popular, and certainly the most suitable. In accordance with this we may define the intent or connotation of a name as consisting of the attributes 'commonly understood to be implied by the name'. Of course this account makes an assumption as to

general agreement which we know does not in fact exist. Hardly any two people do, as a matter of fact, altogether agree as to the attributes implied even by names in common use. Some such assumption however has to be made if Logic is to be treated as a science: without it we could not speak of the definition of a name, nor recognize any tenable distinction between accidental and essential qualities. There is nothing, it may be remarked, peculiarly arbitrary or unreasonable in such an assumption for the purposes of ordinary logic. It is really an assumption of much the same kind as has to be made in all applied sciences, for our definitions and theories seldom or never accurately square with observed facts.

In speaking thus of intension and extension as if they could be regarded apart, we must of course remember, not merely that the existence of each (generally speaking) postulates the existence of the other, but that this existence is actually more or less in evidence, and can only be neglected by an act of conscious abstraction. That is, we cannot help knowing, in the one case, that a class, actual or potential, corresponds to each group of attributes, just as in the other we know that each class was, or might be, determined by its distinctive attributes. But in each case alike we can, for the purpose of discussion, consider one aspect alone. Hitherto we have had before us a set of classes, and we did not trouble ourselves to take account of the attributes by which they were to be distinguished. Now we are supposed to have before us a set of attributes, and we propose, so far as this is possible, to discuss their relations to each other as they stand.

If the reader will recall the investigations of the first chapter, he will remember that the question discussed in respect of extension was this:—When we confine our

attention to the classes denoted by the names, in how many distinct relations of exclusion and inclusion can two such classes stand towards each other? We saw that the number of such relations was five. The questions then followed at once, What is the best verbal statement of the corresponding propositions? and how do these stand in connexion with the propositions of common life and the traditional logic?

The corresponding question here would be, Given two groups of attributes, each representing the intension of a concept or term, in how many distinct relations can these stand towards each other? In saying this it must be remarked that the mutual relation of two pure concepts is in itself a matter of comparatively small importance, for we very seldom think or reason strictly in terms of intension. The question which every one will want to have answered is, how do these relations present themselves when expressed in propositions? Intensive relations can at best yield possibilities in the region of fact: extensive relations are generally understood to yield actualities. A mere classification therefore of the various intensive relations, though it offers a close parallel to that of the extensive, is of small use in itself. We must see them converted into propositions. with the notions as subject and predicate, before we can feel satisfied. What then are these relations? At first sight it may appear as if we were only asking the same question over again; for a group of attributes is itself a class of a certain kind. But reflection soon shows that we are dealing with a subject matter which is, in one respect very different. One group of attributes may indeed include or exclude another, wholly or partially, just as concrete objects may do: but can one such group *coincide* with another, and yet be recognized as a distinct entity from it? I think not; and

therefore the frequency with which such a relation is verbally admitted by those who speak in Conceptualist language seems to show that this view is not carried out with rigorous consistency[1]. There is, surely, a broad distinction between extension and intension in this respect. In the former category, coincidence is admissible; for, though the individuals are identical, we may suppose them to be called up by different names, or by reference to different attributes. Hence, 'All A is all B' is by no means necessarily an identical proposition. But when A and B are groups of attributes coincidence merges into absolute identity; for, as the attribute is a result of abstraction, all distinguishing characteristics have been stripped off. Instead of an 'All A is all B' proposition we get an 'All A is all A'. At most any difference which can be admitted is of a merely verbal nature: a substitution of one name for another. Something much more than this could be extracted from the corresponding propositions in extension.

As regards the relation of inclusion, the case seems plain enough. Suppose two concepts, consisting respectively of the attributes a and $a+b$, and suppose the corresponding class terms to be X and Y. It is clear that every object denoted by the latter name will also be denoted by the

[1] For instance, many German logicians refer to this relation under the name of "Wechselbegriffe", the *Begriff* being composed of *Merkmale* and being therefore by rights abstract. But what they actually understand by them are generally two concepts whose extension happens to be the same, whatever their marks may be. Beneke, *e.g.* plainly says so, "In Hinsicht auf den Umfang nennt man die Begriffe Wechselbegriffe wenn sie den gleichen Umfang haben" (*Logik*, p. 28). Some again more consistently, as I should say, deny the possibility of equivalent, as distinguished from identical concepts. Thus Bachmann, "Wer in B nicht mehr und nicht weniger denkt als A, der denkt eben A; und B, das genau dieselben Merkmale hat wie A, ist eben A, weil es von diesem durch nichts unterschieden werden kann" (*Logik*, p. 111).

former, so that we have the familiar 'All Y is X'; just as, had the relation been the other way, we should have had, 'All X is Y'. But even here a difference must be noted. What we thus obtain is not a quantified predicate, 'All Y is some (but not all) X', but the ordinary form 'All Y is X'. As regards the attributes themselves, one group is a part (*only*) of the other; but it does not follow that, as regards the objects denoted, one class will not coincide with the other. We must not attempt to be more precise than the ordinary A proposition warrants.

The two inclusions just mentioned can be exhibited in a purely formal manner, but we find a difference when we come to the next case. Take that of partial inclusion, by supposing that the concepts may be represented by a, b, c, d, and c, d, e, f: the corresponding verbal terms being, say, X and Y. Material considerations apart—it seems that *any* relation between X and Y as subject and predicate is possible here. No doubt a relation can be deduced, if we modify the subject or predicate by abstraction. Strip off a and b, and we have only c and d left, which, *quâ* extension, include c, d, e, f. That is, we may frame the derivative proposition, 'All Y is X:—if from X are abstracted a and b'. And similarly with the converse relation between X and Y. But clearly such propositions are not what was contemplated, for they have not simply X and Y as subject and predicate.

So again with the relation of total disparateness. If we take the groups a, b, c, d, and e, f, g, h, and suppose the corresponding terms to be X and Y, we cannot claim even a presumption in favour of the proposition 'No X is Y'. Thus, for example, 'Existent marsupial', and 'large quadruped indigenous to Australasia' comprise very distinct groups of attributes, but denote approximately identical groups of

animals. But, on the other hand, 'men under thirty', and 'men over thirty' have every essential attribute in common, but denote totally distinct classes[1].

The last example may suggest that the expression 'material considerations apart' used some lines back, needs qualification. It is quite possible that letter symbols which, as they stand, have no apparent relation to each other, may yet. as representing significant attributes, be brought within mutual reach. It may happen, for instance, that e or f may be distinctly incompatible with a or b, either by their very form, or by obvious and necessary inference. In such a case we might fairly indicate the hostility by a slight change in our letter symbols, as by writing (say) $e'\bar{a}$ for e, and $f'\bar{b}$ for f. It would then be plain that we had the materials for a universal negative before us; for a single pair of incompatible attributes, existing one in each of two otherwise identical concept groups, is sufficient to cause total mutual exclusion of the classes thus denoted. Except however in such a case as this, which must on the whole be rare, it does not appear that the mutual relation of these disparate groups of attributes in any way determines the mutual relations of the classes of things denoted by them; or consequently the nature of the opposition between the propositions into which the terms enter as subject and predicate. Any of the four recognised relations is possible here.

It appears to me therefore that if by 'reasoning in intension' we propose to confine ourselves to those attributes which are commonly understood to be implied by the name, and, starting with two such groups, to examine what sort of

[1] It was noticed, in 1685, by James Bernoulli,—perhaps the first to employ the signs (+) and (−) for this purpose,—that, in a negative proposition, it was not necessary that no qualities in the subject and predicate should exist in common.

propositional forms can be elicited from them, we shall only see our way to the relation of total inclusion. That is, we can (generally speaking) only deal with the universal affirmative in one of the forms 'All X is Y', 'All Y is X'. And, it may be added, such propositions will in strictness be confined to the class commonly described as necessary, essential, or analytic.

This being so, it hardly seems worth while to go further, by attempting to work out a symbolic system with such an extremely defective stock of propositions. Still, if only for historic interest, it will be well to enquire how the above simple relations may best be expressed in symbols.

What sort of symbols then would be appropriate to mark the connection of the attributes in these groups which constitute the intension? On this point there has been comparatively little difference of opinion. It is, in fact, so natural to talk of 'adding on' one attribute to another,—a glance at the Porphyrian Tree seems to suggest this expression,—that very little divergence has existed amongst such logicians as have made use of symbols for this purpose. They have naturally adopted the sign (+); and when they wanted to indicate that 'man' could be defined as 'rational animal', they put the sign of addition between the letters which stood for rationality and animality. Even Hamilton permitted himself to go so far as this: "The concept, as a unity, is equal to the characters taken together, $Z = a + b + c$" (*Logic*, I. 80). And he has had companions here amongst those who had but little general sympathy with anything resembling a mathematical dress for logical forms[1]. Several

[1] E.g. Twesten (*Logik*, p. 211), who has also made use of the *minus* sign for abstraction. This was apparently the only employment of these signs which had come under Trendelenburg's notice, and forms the ground of his objection to the introduction of such expressions into

earlier instances of the employment of this expression have been already pointed out in various notes in the foregoing chapters.

The employment of the inverse sign to that of addition seems equally natural for the purpose of taking off attributes, when we proceed in the other direction along the predicamental tree. When we wish to subduct 'rationality' from 'humanity', and thus to leave 'animality', what can seem more suitable than to make use of the sign *minus?* And accordingly it has been occasionally put to this use since the time of Leibnitz[1].

The reader must however carefully note what a very different signification is involved in the use of these symbols, according as we employ them to connect classes together directly, or indirectly through their attributes. To indicate a class by adding attributes together, is to confine ourselves to such a group of objects as possess both sets of characteristics: and this is a process which we have hitherto represented by the multiplication sign. 'Extensive multiplication' in fact corresponds[2] to 'intensive addition'.

Logic (*Logische Untersuchungen*, I. 20). De Morgan is of an opposite opinion as to the propriety of usage here. He says (*Camb. Phil. Trans.*, x. 192), "The logicians say that a concept is the *sum* of the attributes which it comprehends; the mathematician easily corrects them." He considers the multiplication sign as the correct one to adopt.

[1] Perhaps the earliest employment of the sign $(-)$ for this purpose is by James Bernoulli in his *Parallelismus ratiocinii logici et algebraici* (1685), "Si a conceptu ideæ magis compositæ conceptum minus compositæ auferas, relinquitur prioris differentia: ita quia in conceptu hominis, præter animalitatem involvitur rationalitas, sequitur, ablato animalitatis conceptu, relinqui rationalitatem ceu differentiam. Pariter, relinquitur utriusque differentia, quæ indicatur signo $(-)$, ut $a-b$ significat differentiam inter a et b" (*Op.* I. 214).

[2] It may interest the mathematician to remark that 'addition' of attributes presents a certain analogy to that of powers or logarithms. It corresponds to 'multiplication' of extensive relations.

Similarly to 'subtract' one attribute from a group is to perform the logical operation of *abstraction*. As the reader will remember, we pointed out that the recognized logical abstraction, when somewhat modified so as to be brought into harmony with our extended view of class interpretation, corresponds to the process which *we* represent by the division sign. We indicate it by $X \div Y$ rather than by $X - Y$. This was, in fact, exactly the use which Leibnitz made of the symbols in a passage which I have quoted in the Introduction, when he employed the expression 'man' – 'rational' = 'brute'.

Is there then no way, it may be asked, of so connecting two or more attributes that the result shall be the mere *aggregation* of the corresponding classes? If X and Y stand for the classes marked out by the attributes a and b, how should we simply indicate that aggregate of X and Y which we express by $X + Y$? What we want, apparently, is a sign for mere alternation or disjunction. We wish to denote the class of things which possess either attribute a or attribute b, but to do this indirectly by putting some connecting symbol between a and b. We might resolve to indicate this by some such sign as, say, $(\sim)$, so that $a \sim b$ should represent the aggregation or 'addition' of the corresponding classes X and Y. Such a symbol would however seem to stand alone, unbalanced by any similar inverse symbol indicating the exception or omission of one class from another. We might, of course, interpose some symbol between a and b, and interpret it as indicating the exception of the class corresponding to b from that corresponding to a; but surely this would be scarcely more than a subterfuge. What we should thus be representing would really be class relations pure and simple, not relations of attributes. As a matter of fact, I believe, symbols of this latter description have not

been introduced into Logic. So far as resort has been made to any such aid, it has been attempted to carry the work through with the signs of addition and subtraction only, or with some kind of arbitrary symbols precisely equivalent to them.

It does not seem to me therefore that an intensive interpretation, strictly carried out, would lend itself to any but universal propositions, nor even to these except when essential or verbal. To affirmatives of this description it applies readily enough, and by a slight device it can be made to apply also to the negatives. But what are we to do in the case of accidental, and particular, propositions? As regards the former, the difficulty perhaps is not so great. Suppose, for instance, any one makes the following assertion, All native American citizens know the name of their president. He can hardly be said to be adding on the latter attribute to the intension of the term Americans; for no one supposes that we are going to alter or enlarge the logical meaning of this term:—unless indeed, with Professor Bain, we so enlarge the range of the essential attributes as to make them include every known property of the things denoted by the name. What we must be supposed to do, in accidental predication, is to be making a secondary class of attributes which are to be associated[1] with, but not incorporated into, the smaller class constituting the essence:—a sort of Uncovenanted Service without the permanent rights of the regular members. Probably this view is more in accordance with popular thought than with the strict definitions of the logicians.

But the case of particular propositions is on a very

[1] "Whatever does not form part of the comprehension of the concept or of the signification of its name, is not *part of*, but *joined to* the essence" (Mansel's *Aldrich*, p. 33).

different footing, and it seems impossible to find a satisfactory explanation of these without simply falling back upon an extensive interpretation, by appealing to individual examples. Mansel himself,—the most consistent of English Conceptualists,—admits that 'Some X is Y' must be understood as declaring that in certain actual cases the distinguishing attributes are found to coincide; and what more could one say when interpreting in pure extension[1]?

On the whole then it seems that a consistent intensive interpretation would find an opening for the symbols + and −, but that it would use them for the purpose of expressing,—approximately only, in the case of the latter,—operations which we have indicated by × and ÷. And as to those operations which *we* denote by + and −, it might possibly find a place for the first though I do not think that it actually has done so.

It need hardly be remarked that this is not the course which has been actually adopted. Logicians have mostly consulted convenience at the expense of rigid consistency; and when they have employed these four arithmetical symbols they have constantly done so in a way more in harmony with that adopted in this work. Hence their language is often tinged, and their usage modified, especially in the case of the older symbolists, in a way which it would be difficult to understand unless we keep in mind the characteristics of the system which they nominally professed[2].

[1] Castillon, whose views are discussed further on, draws a distinction between those particulars which are the converse of universals (with which alone he practically deals) and those which are true "parce que quelques A sont B et quelques A non":—i.e. those understood in the customary sense, as being founded on the mutual relations of individual A's and B's.

[2] One example may be given here from Lambert: "Die Redensart, 'A so nicht m ist,' wird so gezeichnet $\frac{A}{m}$, weil nun m von A als eine Modification von ihrer Substanz kann ab-

The reader will find plenty of illustrations in support of this remark in the historical notes of the next chapter.

The foregoing discussion, though entirely re-written, is in substantial agreement with that of the first edition. At that time I had been unable to meet with any really thoroughgoing attempt to carry through the intensive interpretation, and to represent the results in symbolic language. Not long afterwards I succeeded in finding an essay which I had for some time been in search of, viz. the *Nouvel Algorithme Logique*[1] of G. F. Castillon. It is so nearly unique in its treatment, and so closely carries out what I had thought would be the path of rigid consistency on the principles which it adopts, that I give a brief summary of it here.

Castillon's scheme is one of pure intension. Let S stand for a subject concept; A for one of the component attributes which we propose to employ as predicate; and let M represent the remaining attributes. Then $S = A + M$ will represent the Proposition, All S is A. Thus 'humanity' is composed of 'rationality' *plus* a number of other attributes summed up in 'animality'. The above denotes a universal affirmative. For the universal negative he writes $S = -A + M$.

strahirt werden." This is a purely intensive interpretation; whereas *we* should proceed to represent 'A which is not m' extensively, in some such way as $A - Am$ or $A - m$. Elsewhere Lambert himself has used this subtractive sign very nearly as we do.

[1] Read at the Berlin Academy, Nov. 3, 1803. It may be regarded as the continuation of a paper read before the same society in 1802. In this latter, starting with the four relations of two concepts (i.e. omitting, quite consistently, that of coincidence) he deduces the sixteen consequent relations between three terms, and compares his results with those of the syllogism. His conclusions seem to me open to criticism on other grounds than those of his own principles, but they represent a thoroughgoing attempt at the intensive interpretation. Thus the inclusion of A in B, and of B in C, yields rightly the conclusion 'All C is A', not 'All A is C'. Castillon states that he was first led to these enquiries by reading the correspondence between Lambert and Holland.

Here comes in a decided awkwardness which leads (in my opinion) to some confusion. When we employ a negative sign in an equation, it is almost impossible not to want to transfer it to the other side, and change it into +; and this is what Castillon sometimes does undertake to do. But it is surely clear that though we may talk of a 'negative attribute' we mean something quite distinct from the subtractive process commonly indicated by the negative sign[1]. When we deny 'immortality' of all men, we are not merely subducting an attribute from one side of an equation. It would have been better therefore to have used some such sign as $\bar{A}$. Apart however from this fault of expression, the equation $S = -A + M$ means that the concept S is built up of an attribute $(-A)$ which denies, *plus* a remainder of other attributes. Thus, 'No man is immortal' means that the attributes of 'man' are composed of one which is negative, or which denies 'mortality', and of a number of others which are, as before, positive in their character.

There are several points which deserve notice here. In the first place it will be seen that so far we are dealing only with *essential* propositions. This seems inevitable on such a scheme. Castillon does indicate a notation (by use of small letters, instead of capitals) to stand for accidental qualities; but he makes no further use of it, nor would it be quite consistent to do so. Again, when we look only to the concept, what we have before us is a *unity*, viz. a group of attributes bound up into a whole. The plurality lies in the individuals comprising the extension or denotation: it is only in collective propositions that we can be said to make a unity of these. Accordingly we are told in plain terms:—"Je crois donc pouvoir regarder comme vrai ce que mon

[1] The confusion involved here was pointed out in a former chapter (see page 57).

algorithme logique prouve par la simple inspection, que tout jugement est naturellement singulier, ne présente tout au plus que la notion d'unité jointe à celle du sujet. Ou plutôt, que tout jugement est un, relativement à la quantité". "L'expression $S = A + M$, qui représente bien exactement un jugement affirmatif, n'entraîne aucune idée d'universalité ou de particularité; elle présente tout uniment le sujet S comprenant l'attribut A".

But where his consistency is most shown is in the treatment of particular propositions. These have always been somewhat a trouble on this view, and even such a writer as Mansel is forced to admit that when we assert 'Some X is Y' we certainly mean to say that some of the concrete individuals denoted by the term X do possess the attribute Y: in other words, to fall back upon a simple 'extensive' fact. Castillon gets out of the difficulty by maintaining that the only real particular judgment (he draws a distinction, which we need not discuss, between the real and 'illusory') is that which is obtained from a universal. "Tout jugement affirmatif particulier est donc au fond, et précisément parlant, un jugement universel, dans lequel l'attribut (le concept le plus simple, le concept abstrait ou composant), a été mis à la place du sujet (à la place du concept le plus composé)". Thus from 'All men are rational' we derive in the ordinary way, 'Some rationals are men'. Put into symbols this stands, $S = A + M$, therefore (by subtraction) $A = S - M$. That is, abstract 'rational' from 'man', and we have the difference between the concepts 'man' and 'animal', yielding a particular proposition. "Cette restriction, ou marque de particularité, se réduit au fond à abstraire du concept S toutes les marques qu'il comprend de plus qu'A, marques dont l'assemblage est indiqué par l'indéterminée M". As a consequence he admits the fol-

lowing syllogism:—"quelque A est B, quelque B est C, donc quelque A est C, lequel ne se trouve dans aucun mode d'aucune des quatres figures de syllogismes,...mais qui n'est pas moins bon en lui-même pourvu que les propositions particulières soient vraies comme elles le sont ici, parce qu'elles sont les converses accidentelles des universelles". (Paper of 1802.) In this case it is seen that the sign for subtraction means,—as it surely ought when applied to intension,—not subduction or exception of a contained from a containing class, but abstraction of an attribute from a group of attributes. As Castillon says, "le signe + pour indiquer la synthèse, le signe (−) pour indiquer l'analyse". This use of the signs is illustrated by its application to the doctrine of Definition. "$O = A + B + C + D$ fournit la définition de l'objet, le concept correspondant à l'objet...: $O - A = B + C + D$ fournit le concept de l'espèce sous laquelle sont compris les objets O: $O - A - B = C + D$ fournit de même le concept du genre".

It need hardly be added that, on these principles, the distinction between categorical and hypothetical is rejected; and that the relation of greater and less in respect of the subject and predicate terms is inverted: that is, in 'All X is Y' it is X which contains Y, not Y which contains X, as in ordinary predication.

As we are only concerned with the doctrines of this writer in so far as they serve to illustrate what appears to be the rigidly consistent outcome of a certain logical principle, no detailed criticism is here offered. I simply add his symbolic formularization of *Barbara* and *Celarent* in order to illustrate his consistency of treatment,

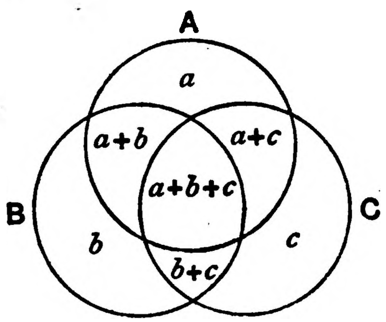
$$\begin{cases} M = A + N \\ S = M + P \end{cases} \qquad \begin{cases} M = -A + N \\ S = \quad M + P \end{cases}$$

$$\therefore S = A + N + P \quad \therefore S = -A + N + P.$$

In the above investigation we have started from the customary logical point of view, by regarding the intension of a term as comprising a strictly limited number of attributes; and our conclusions have been modified accordingly. It will be convenient now to examine the question from a somewhat different point of view.

Instead then of considering the intension as comprising a somewhat arbitrary selection of attributes, what would follow if we consider it to comprise the sum-total of all the attributes possessed exclusively by the members of a class? It is difficult to work this view out completely, for attributes are flimsy things to deal with; but so far as I can see my way we appear to be thus led to a scheme very much resembling the class-view which has mainly occupied us throughout this volume.

For simplicity, let us suppose three attributes only, *a*, *b*, *c*; and let *A*, *B*, *C*, be the classes which possess these attributes: that is, *A* is to comprise exclusively those objects which possess *a*, and similarly with *B* and *C*. We may illustrate by the following diagram,—



We have, that is, a scheme of eight compartments, precisely as we had on the former interpretation.

First as regards negative attributes; the case seems much the same as on the common view. If *A* includes everything which possesses the attribute *a*, then all that lies

outside A does not possess that attribute, and may be conceived therefore to possess the contradictory attribute. Of course this is not quite the conventional view, for I think we are far more ready to construct in this way negative classes than negative attributes. Though it is not in accordance with conventional thought to construct a class of 'not-long things,' no violence is done to familiar association by such a construction: but it would be a very different thing to propose 'not-length' as an attribute covering that same class. As a rule we simply drop the attribute, and only in very rare cases construct a true negative one. Immortality and incorruptibility, for instance, may be regarded as standing to mortality and corruptibility in much the same way as immortal and incorruptible stand to mortal and corruptible. That is, they are true negative attributes, covering all things, within the range of discussion, which are not covered by the positive attributes. What popular thought admits in these exceptional cases we must, I think, claim in all cases.

The two main characteristics of the intensive, as contrasted with the extensive interpretation, are readily displayed in such a figure. In the first place, the more the intension, the less the extension, and conversely. The class ABC is the smallest in extent, but its members possess the greatest number of attributes, viz. a, b, c. Again, the included and the including members change places according as we read the proposition one way or the other. Extensively regarded, the class AB is included in A; but intensively regarded the attribute a is included in $a+b$.

The question of existence, which, when applied to our class interpretation, gave rise to considerable discussion, occupies a much less disputable position here. The mere fact that an attribute is actually conceived, is proof of its

existence; for its origin and its home are in the mind. To speak of a class, and then to deny that there are any objects in it, comes very near, in popular estimation, to a direct contradiction: but to speak of an attribute, and only to deny that there are any objects which possess it, is at worst regarded as misleading. The natural way therefore of regarding such a diagram as the above is not to consider it (as we did before) as a mere framework of possibilities waiting for propositions to curtail it to actualities. We should rather regard it as a scheme of intensive propositions as well as of attributes. Each compartment is taken for granted to be tenanted by an occupant,—ideal perhaps, but none the less fitted on that account to give ground to ideal propositions.

A glance at the above figure will, I think, show the propriety of that transfer of the additive sign which was discussed a few pages back. As was shown in the second chapter, the sign (+) almost inevitably comes to hand in order to denote what is covered by one or other of the two classes A and B; whilst the sign (×) serves with reasonable propriety to denote what is common to the same two classes. But when we turn to the other aspect, we see that in describing intensively the class AB we almost instinctively say that it is marked out by possession of the attributes $a+b$. Then comes the further question, how are the attributes to be connected when we indicate the class $A+B$? The answer seems to be that they are connected by mere alternation or disjunction. That is, we should indicate the class AB by saying that its members possess "attributes *a and b*", and class $A+B$ by saying that its members possess "attributes *a or b*". In consistency with this, the subtractive sign, when applied to the intension, seems necessarily to correspond not to logical subduction or ex-

ception but to abstraction. Thus $(a+b)-b$ is the withdrawal of b from $a+b$; but this carries us from class AB to class A, which is universally recognized as being a step of the nature of logical abstraction,—and therefore approximately the analogue of the inverse of our logical multiplication as described in Chap. II.

Take the familiar A, I, E, O propositions in order. It is clear that a is included in $a+b$; or, put in the class form, All AB is A. Similarly we have, All ABC is BC; and so on. That is, we readily enough obtain a series of universal propositions in this way. They are all affirmative, and, if the attributes were really, in the common sense, part of the "meaning", they would all be essential also. The converses of these give us a series of particular propositions, of the form 'Some (only) A is AB'.

In saying this it is, of course, taken for granted that each class, or subdivision, has occupants: in other words, that every combination of attributes is represented in existence. If this assumption is not admitted we must then couch all these affirmatives in the hypothetical form.

As regards negative propositions, the problem assumes a decidedly different aspect from that which it presented before. We then saw that a mere contemplation of the two groups of attributes,—supposing that no one of the individuals composing these groups is formally contradictory of another,—gives no ground for or against the supposition that the corresponding classes are exclusive of each other. But when, as here, the enumeration of attributes is supposed to be exhaustive, it is otherwise. Thus the subdivisions marked a and b are mutually exclusive, for, by supposition, every member common to A and B stands in the subdivision marked $a+b$. The corresponding class rendering is given by the proposition, No $A\bar{B}$ is AB. In saying this it is, of

course, not assumed that a and b, as attributes or groups of attributes, are mutually exclusive. On the contrary, they may have, and probably will have, a substratum in common. Similarly the distinct subdivisions $a+b$ and $a+c$ give us the negative, No $AB\bar{C}$ is $A\bar{B}C$. We may thus go through every pair of subdivisions. The particular negative here, it will be seen, is not derived, as in common logic, from the universal, but is the complementary part of the particular affirmative. Since some (only) of the a's were $a+b$ it follows that some (only) of them are not $a+b$. That is, of the subdivisions marked a and $a+b$, each is a part, and a part only, of the class A. They yield the pair of particulars, Some (only) A is AB, some (only) A is not AB. If we want the 'Some A' of the common logic we must deduce it directly from the universal, by saying, 'Some $A\bar{B}$ is not $\bar{A}B$':—a rather useless expression, since it is obvious, by the symbolic form, that the universal holds good.

The general outcome of this discussion seems to be that when we take the extreme view,—viz. that of identifying the intension with the sum total of the distinguishing attributes, and regard the corresponding classes as hypothetical only—we obtain something very much resembling our original system. The substratum of class relations is the same, but the symbolic notation is far more awkward. It seems decidedly more convenient, and more nearly in accordance with popular thought, to make our reference to the classes a direct one, instead of an indirect one through the medium of the distinguishing attributes.

In the preceding pages I have worked out what seems to me the consistent or thoroughgoing development of the intensive interpretation, so as to set it against the corresponding development of the extensive interpretation adopted in this volume. It need hardly be said that con-

sistency of this type is the aim of the logician, not the attainment of the popular thinker. In practice the two sides of the term or proposition are constantly intermingled, and each may in turn be determined by the other. But for systematic purposes it is quite reasonable to consider each side by itself, without introducing the question how far the other side had been employed in its determination. Thus in our procedure hitherto we have taken it for granted that the classes were placed before us in their mutual relations of extension; and we proceeded onwards from this point, without going back to ask whether, and to what amount, the intensive side had been appealed to in determining these relations. Just so on the plan described in this chapter. It is taken for granted that what is placed before us is the mutual relations of two intensions, or groups of attributes. This is our starting point, and we proceed onwards from this, without going back to enquire whether these relations had actually been presented to us by the aid of specified individuals, or whole classes.

I think that this is the sense in which the 'intensive' rendering has generally been understood. It is certainly that in which Hamilton understood it,—as see his parallel versions of the syllogistic process, in his *Logic*,—and which was carried out with such consistency by Castillon. It also seems to me approximately the sense in which the same rendering was understood by Leibnitz, by Lambert and by others.

I have never contended that either of these extreme views represented the common process of ordinary thought, nor that either should be made the basis of the rules of ordinary Logic. All that I maintain is that the class interpretation,—and this remember, as interpreted and generalized in the last chapter,—seems the best for working out such

extensions of the reasoning processes as form the scope of the Symbolic Logic.

Intermediate interpretations are, of course, quite possible; and, as our simple symbolic rules are very flexible so far as their application is concerned, it will be found easy to fit them in with such renderings. The most carefully worked out suggestion that I have seen in this direction, compendiously stated as it is, appears to be one by Dr E. G. Husserl[1]. He claims it to be a purely intensive rendering of our judgments (ein reiner Inhaltsurtheil): to me it seems distinctly to occupy an intermediate position, but to incline more to the extensive side than to the intensive. On this view the version adopted for the Universal affirmative is this:—An object of concept S is, as such, an object of concept P; or, So far as anything has the marks of S it has those of P. As Dr Husserl points out, this interpretation is substantially the same as that which is familiar to English readers from its adoption by Mill. It is quite true that the reference to the concept element here is decidedly more prominent than when we talk of 'Every S', or of 'The class S;' and several of the detailed explanations fit in better with the view which it claims to represent. Thus, for instance, as regards the 'multiplicative' process, "By AB we understand the object of the concept which embraces the marks of both A and B and of these alone: that is, the *sum* of their marks." So again, "By $A + B$ we understand the object of the concept 'either A or B'". As the reader will remember, we have had to take notice of both these particular forms, and have recognized that they assume more of the intensive than of the extensive aspect. On the other hand, what I regard as the most characteristic feature of the

[1] *Gött. Gel. Anzeigen*, 1891.

true intensive rendering, viz. the inverse relation of subject and predicate in respect of inclusion, is missing. As we have seen, a very different account was given by Hamilton, Castillon, and others, of this particular relation. With Dr Husserl, as with Mill, it is the predicate, not the subject, of the universal affirmative, which includes the other. In saying this, the reader will understand that I am not in any way objecting to the scheme of interpretation in question, which seems to me well worked out, but only to its claim to rank as a true intensive interpretation. Its author has shown that we can put an explanation on all the fundamental rules of operation by adopting this plan; though in respect of some of his explanations (e.g. those of the symbols 1 and 0, discussed above, on page 438) I cannot find myself in agreement with him.

CHAPTER XX.

HISTORIC NOTES.

I. *On the various Notations adopted for expressing the Common Propositions of Logic.*

ATTENTION has been already repeatedly called to the general fact of the great variety of symbolic forms which have been proposed from time to time by various writers, but probably few persons have any adequate conception of the extent to which this fertility of invention has been carried. I have therefore thought it advisable to throw together, into one tabular list, the principal forms, so far as I have observed them, in which one and the same proposition has thus been variously expressed. The reader will of course understand that it is not here attempted to give a critical account of the schemes of the various authors referred to. All that is proposed is to indicate the principal different points of view which have been selected. It may be added that this table has not been drawn up with a mere wish to make a collection :—had this been so more specimens might easily have

been included. Most of these forms, it must be remembered, have been made the instruments of a more or less systematic exposition of the subject. In so far therefore as the notation is not entirely arbitrary,—which it very seldom is,—we shall find it instructive to compare the different aspects of the same operation to which they respectively direct attention. In no other way, I think, is it possible to realize how many-sided is the aspect of the familiar predication of ordinary life and of the common logic. Nothing is more surprising to me than the confidence with which it is often laid down that this or that shade of signification is what people mean, or logicians hold, as to the import of propositions.

For the purpose of this investigation the Universal Negative has been selected, as being about the simplest and least ambiguous of all forms of statement. For convenience of reference and comparison the propositions are expressed in the same letters in each case, S and P standing respectively for the subject and predicate of the original proposition: viz. No S is P.

The analysis by which we reach these various forms is as follows:—

I. In the first place we may regard the proposition under its *existential* aspect; and this in two different ways, one negative and the other positive. In the former case what we do is to deny the existence of a certain class of things, viz. of those which are both S and P: a denial which may be either absolute and unrestricted, or confined to a certain indicated range. In the latter case we take account not of SP, but of its contradictory or complementary class ($\overline{S} + \overline{P}$, here), and declare that this class exhausts the universe.

II. We may cast the proposition into the form of an *identity*. What we then do is to introduce a negative predi-

cate, making the terms of the proposition respectively S and not-P, and identifying the former with some undetermined part of the latter. There are many slightly different ways of indicating this, but in all alike the most appropriate form of expression is that of an equation; and, as a matter of fact, the mathematical sign (=) has been most commonly used for the purpose. Indeed this sign ought in strictness to be reserved for this particular interpretation: it is somewhat of an abuse to employ it, as is too often done by non-mathematical logicians, to indicate mere predication.

III. A very distinct modification of the above view may be described as that of *subsumption*. Instead of regarding S as identical with a portion of not-P, we may prefer to say that it lies somewhere within that range: that it is included in not-P. So regarding the relation, we should in consistency reject the sign of equality, and employ instead either the mathematical symbol of inclusion ($<$), or some modification or analogue of this adapted for logical use. This interpretation probably keeps somewhat closer than the last to the popular account of the matter, and seems at present to be the more prevalent symbolic rendering.

IV. Again, we may consider the proposition as expressing the *mutual exclusion* of the two elements, subject and predicate. This account obviously applies in a very direct manner to the Universal Negative; and by the introduction of negative terms it may be applied to affirmative propositions. It may also,—by the employment of a sign to stand for exclusion which is not entire, but partial only,—be extended to the case of particular propositions. As the relation in this case is a convertible one, some symmetrical symbol becomes appropriate here.

V. Another plan is to regard the proposition as expressing a consequence, or *implication*: If S then not-P, or, the

presence of S implies the absence of P. This interpretation seems to me to involve a distinct step from the objective rendering of Logic towards the subjective; for, whereas Subsumption has regard rather to the actual class relation itself, Implication introduces the attitude of the mind towards that relation in the way of an inference to be drawn. The relation in this case is not convertible, any more than in (III.), for it does not follow that the absence of P implies the presence of S. Accordingly some kind of unsymmetrical symbol becomes appropriate to represent the copula.

Under this general heading a well-marked variety may be introduced by making a whole proposition in itself the element of implication, that is, by letting S and P stand, not for simple terms or marks, but for entire statements, or for the truth of such statements.

VI. Again, we may resolve to keep more closely to the common or *predicational* expression of the proposition, regarding it as an indication of a relation, not between S and not-P, but between S and P. This relation is convertible, and we should therefore naturally seek for some symmetrical symbol; that is, for a symbol which by its own form shall suggest that the mutual relations of the subject to the predicate, and of the predicate to the subject, are the same.

Under this rather wide and miscellaneous heading must be placed, not only certain logical innovations, such as schemes for the Quantification of the Predicate, but also all those renderings which simply repeat the familiar logical processes under a slight algebraic disguise.

VII. Lastly, we may interpret the proposition in its conceptualist or *notional* rendering; regarding S and P as being directly representative, not of classes of things, but of attributes or groups of attributes.

Grouping them in accordance with this scheme, we may arrange our species as follows:

I. Existential				
	(1) Negative	1.	$SP=0$	Boole.
		2.	$S(0=P)$	Macfarlane.
	(2) Affirmative	3.	$(\bar{S}+\bar{P})_1$	Mitchell.
II. Identity		4.	$S=v(1-P)$	Boole.
		5.	$S=\frac{0}{0}(1-P)$	Boole.
		6.	$S=v(X-P)$	Wundt.
		7.	$S=Sp$	Jevons.
		8.	$S=1-P-y$	Delboeuf, Murphy.
		9.	$S=\frac{P}{\infty}$	Holland.
III. Subsumption		10.	$S<X-P$	Drobisch.
		11.	$S<-P$	Segner.
		12.	$S \angle \bar{P}$	R. Grassmann.
		13.	$S \prec \bar{P}$	Peirce.
		14.	$S \underline{\notin} P_1$	Schröder.
IV. Mutual Exclusion		15.	$S \overline{\vee} P$	Ladd-Franklin.
V. Implication		16.	$+S-P$	Darjes.
		17.	$S : P'$	McColl.
		18.	$S \frown$ not-P	Pokorny.
		19.		Frege.
VI. Predicational		20.	$Sx=-P$	Maimon.
		21.	$S=-P$	Victorin.
		22.	$S\,.\,P$	De Morgan.
		23.	$S\ )(\ P$	Wundt.
		24.	$S-nP$	Jäger, Lichtenfels.
		25.	$S>P$	Ploucquet.
		26.	$nS-P$	Ploucquet.
		27.	$tS \parallel tP$	Bentham.
		28.	S :  : P	Hamilton.
VII. Notional		29.	$L-S \infty P$	Leibnitz.
		30.	$\frac{S}{m}=\frac{P}{n}$	Lambert.
		31.	$S>\frac{P}{n}$	Lambert.
		32.	$S=-P+M$	Castillon.
		33.	$S \notin P_1$	Husserl.

1. This form will be thoroughly familiar to every reader of this volume. It simply indicates the destruction of the class SP; or, in other terms, the emptiness of the corresponding compartment in the complete scheme of possibilities. I prefer it myself as one of the primary statements of such a proposition.

2. The characteristic of Dr Macfarlane's notation lies in the determination to indicate the nature of the logical universe in the case in question. He considers that it is the *subject* of the proposition which marks the limits of the implied reference. "Every general proposition refers to a definite universe; which is the subject of the judgment, and, it may be, of a series of judgments. For example, 'all men are mortal' refers to the universe 'men'. 'No men are perfect' refers to the same universe" (*Algebra of Logic*, p. 29). Hence his symbolic form may be read off, Within the universe of S there is no such thing as a P. I have already (Chap. IX. p. 251) given my reasons for dissenting from such an assumption as to the nature of our universe. It seems to me somewhat arbitrary to seek the limits of reference in the subject of the negative proposition rather than in the predicate.

3. This is the plan of Dr Mitchell, as described in the Johns Hopkins *Studies*, and appears to me one of the most ingenious and original modifications of the general Boolian method yet proposed. Starting with the principle that every term, simple or complex, has its contradictory, and that if the former be equated to 0 the latter must be equated to 1, we hereby state that the contradictory of SP exhausts the universe, and write the statement in the form $(\bar{S}+\bar{P})_1$. In such an elementary case there is little or no gain: the merits of the scheme are displayed when we come to deal with complex groups of terms, where the contradictory expression

is sometimes actually the simplest. The scheme leads to many important changes of procedure. For instance, in the combination of several premises, whereas the aggregate import is obtained by *addition* when each was equated to zero, it is obtained by *multiplication* when each is equated to unity. Again, elimination assumes the form of simple omission: e.g. if $xy + yz = 1$, we eliminate y by simply writing $x + z = 1$. Moreover particular propositions can be conveniently treated in combination with universals. If F and G are logical groups of terms, write F_1 for $F = 1$ (interpreted as above) and G_u to indicate that G simply exists, that is, that G comprises an alternation of terms, one or other of which has to be saved; and we obtain a number of formulæ such as,

$$F_1 G_1 = (FG)_1,$$
$$F_u + G_u = (F + G)_u,$$
$$F_u + G_1 -\!\!< (F + G)_u.$$

The method is applied by Dr Mitchell to the solution of some extremely complicated problems.

4—7. These four forms are to all intents and purposes precisely equivalent. The only distinction between them is that (4) introduces an arbitrary sign (v) to express the entire indeterminateness of the selection to be made from not-P; that (5) employs a familiar mathematical symbol to express the same characteristic; whilst (7) disguises the indeterminateness by describing S as the S-part of what is not P, instead of prominently indicating that it is an unknown part. It also abbreviates by substituting a single letter p for the longer equivalent $1 - P$. No. (6) is employed by Wundt (*Logik*, 1880) in his critical account of the symbolic processes. It differs from the fourth and fifth by making use of X as the universe-symbol, instead of 1. As regards

the representation of the class not-P by p, in (7) there is one serious defect. We cannot readily represent the negation of a complex class. The other schemes meet the difficulty. 'What is not both B and C,' and 'What is neither B nor C' can be represented by $1-BC$, and by $(1-B)(1-C)$; or, more briefly, by $\overline{BC}$ and by $\overline{B}\overline{C}$. But on the plan of employing small letters to mark negations, ab would stand for 'What is neither a nor b'. There is no ready mode of simply expressing the negation of AB *as a whole.* We have to express the results of this negation in detail, by writing $a+b$, or using some equivalent expression.

8. This is a form employed by Prof. Delboeuf in his *Logique Algorithmique*, 1877; and seems to me practically identical with one proposed by Mr J. J. Murphy (*Relation of Logic to Language*, 1875: *Mind*, v. 52). The distinction between these, and the preceding four forms, consists in the adoption of the subtractive instead of the multiplicative symbol. Whereas those four say, Make an indeterminate *selection* from not-P and we obtain S; these say, Make an indeterminate *rejection*, by subtraction or omission, and we obtain S. The form is not incorrect, but we must remember that it demands a tacit condition, viz. that y shall be included in $1-P$. When this condition is definitely expressed, we are brought round to Boole's form:—If y is included in $1-P$, so that $y=v(1-P)$, then

$$1-P-y=(1-v)(1-P);$$

but $1-v$ having exactly the same limits of uncertainty as v, this may be written $v(1-P)$. That is, we have

$$S=v(1-P).$$

It should be remarked that Delboeuf divides this general form into several distinct cases according to the extent of the whole universe covered by S and not-P.

A similar expression, though couched in a very defective form, has also been adopted by Dr H. Scheffler, in his elaborate *Naturgesetze* (Part III. 1880). He assigns the general form $a - x = b - y$ to represent the identity, in respect of extension, of the subject and predicate a and b. By varying x and y we can make a and b coincide, or can cause either to include the other. He appears to recognize however that we cannot thus obtain a true negative proposition, the limit yielded by this notation extending only to the unmeaning result, $0 = 0$. Special symbols for negative terms must be employed if we look for success in this direction.

9. This was a scheme proposed by Holland, the friend and correspondent of Lambert. Though not sound in this particular application it deserves notice, both for its ingenuity, and historically as an anticipation of some more recent schemes. (It is given in Lambert's *Deutscher Gelehrter Briefwechsel* (I. 17), where it is however only offered as an "unreifer Gedanke".) The generalized propositional form which he suggested was $\frac{S}{p} = \frac{P}{\pi}$. This is really nothing else than Boole's $vS = v'P$, with the difference that the arbitrary or numerical factor is placed in the denominator instead of in the numerator:—that is, the factors represent arithmetical multiplication by $\frac{1}{p}$ and $\frac{1}{\pi}$, rather than logical division by p and π. There is no proposal of an inverse logical operation here, though Holland had realized this process elsewhere. The consequent restriction as to the range of value of p and π is of course that they must lie between 1 and ∞, just as in Boole's form v and v' lie between 1 and 0. What we express symbolically therefore is, 'Some S is some P', where 'some' may range from *none* to *all*. So far good. Where he goes wrong, as already noticed in Chap. VII., is by interpreting the

limiting case $S=0P$ (viz. that in which $p=1$, $\pi=\infty$) as signifying 'All S is no P', instead of 'All S is *nothing*'. "Wird p oder π unendlich, so ist der Begriff negativ." This error applies to his treatment both of the particular and the universal negative. The fact is that his form is extensible enough to cover particular and universal affirmatives, with either distributed or undistributed predicates; but in order to make it stretch so as to cover negative propositions we must introduce some distinctive symbol for negative terms.

The last six expressions concur in employing the equation symbol ($=$), and appropriately so, for what they represent is the identity of the subject S with some portion of not-P. The five which follow must be considered as belonging to a closely analogous general class, though actually employing a different and somewhat less suitable symbol, viz. ($<$), or some analogue or modification of this intended for logical purposes.

10. This was employed by Drobisch, in the *Logisch-mathematischer Anhang* to the first edition of his *Neue Darstellung der Logik* (1836), but is omitted in later editions. Two points deserve notice here. First as regards the connecting symbol itself. We are familiar with it in mathematics as signifying 'less than': it is here transferred by analogy to the signification 'included in', or 'identical with a part of', and is therefore equivalent in result to the equation symbol when, as in the last examples, this is affixed to a predicate affected by some indeterminate factor. This transfer of the sign ($<$), however convenient, does not seem to me quite accurate, though the signification is clear enough. We need hardly say that as here used[1] it refers to the *extent*

[1] Lambert, who was perhaps, with the exception of Segner, the first to employ these signs ($<$ and $>$) in Logic (as presently pointed out, he employed them *intensively*) takes their propriety for granted: "Die Zeichnung $A>B$ scheint ganz natürlich zu bedeuten, der Begriff A enthalte ausser

not to the *intent* of the terms S and P. Secondly, as regards the predicate term, the notation is curious as showing the great difficulty experienced, by logicians brought up in the old traditions, in realizing the conception of a 'universe' which could be represented as a whole by a single symbol. The letter X does not here stand for really 'all', like our 1, for this would be to introduce an "unendlicher Begriff", or "infinite term", somewhat alien to old association. Drobisch only undertakes symbolically to embrace a finite but uncertain portion of this infinite universe. What he does is to take a class term X of uncertain extent, only demanding that this extent shall be greater than those of S and P together: this may be regarded as finite, and therefore suitable for logical treatment. When our negative predicate, not-P, is thus brought down to finite extent in the form of $X - P$, we can appropriately deal with it as inclusive of S. We signify that S is a portion of not-P; and express this fact by the use, not of an equation formula but of an inclusion formula, as $S < X - P$.

11. This is of considerable interest historically, as Segner's *Specimen Logicæ* (1740) is the first systematic attempt, so far as I have seen, to construct a Symbolic Logic. Nothing apparently had then been produced in this way, beyond a few ingenious suggestions by Leibnitz; and most of these were then unpublished. The sign $<$ is used by Segner, for the same purpose as subsequently by Drobisch, viz. to mark inclusion: $A < B$ indicates that the extent of B is inclusive of that of A. But in one respect he seems

den Merkmalen des B noch mehrere" (*Briefwechsel*, I. 10). There is a much earlier suggestion of this notation,—if indeed it may not be called something more than a suggestion,—in a short logical paper, consisting of heads of theses, by James Bernoulli, in 1685 (*Opera*, I. 214). Hoffbauer (*Analytik der Urtheile und Schlüsse*) and others, have also used the same sign.

to me distinctly in advance of Drobisch, and very much in advance of his time. This is in his free use of negative terms in their fullest extent,—he retains the old name of "infinite" for them,—for the representation of which he uses the negative sign. Thus if A stands for 'man', $-A$ stands for 'not-man'. It may be added that he had fully realized the fact that it is symbolically indifferent whether we employ A to stand for what would commonly be called a positive term, or for its contradictory, provided we preserve the antithesis between $+$ and $-$. Thus A may stand for *non-triangulum*, and then $-A$ will stand for *triangulum*, and so forth. Hence his expression $S < -P$ indicates quite generally that S is extensively a portion of not-P.

This notation is of course somewhat crude, being of no great help even within the narrow limits of the syllogism. The various syllogistic moods are however worked through with its aid, though with certain departures from the common view which need not here be described. It may interest the historical student of this subject to point out that Segner not only describes the symbolic procedure by which from two such premises as $A < B$, $C < D$, we can infer $AC < BD$; but that he also expressly directs attention to what is sometimes called the 'Law of Duality', viz. that $AA = A$:—"subjecti enim idea, cum se ipsa composita, novam ideam producere nequit."

There are a number of other interesting points which must be passed over here for want of space. The work appears to be extremely scarce: the copy which I consulted, viz. that in the Glasgow University Library, was the only one whose existence in this country I could ascertain. It may be remarked that the same notation has been since adopted,—I presume independently,—by G. Hagemann, in his *Logic und Noetik*, 1870; who has applied the same form

to the case of disjunctives, writing $S < B, C, D$ for 'S is either B or C or D'.

12. Robert Grassmann's scheme was published in 1872, under the title of *Begriffslehre oder Logik.* Like a number of other writers on this subject he seems to have worked out his results in entire ignorance of all that had already been done in this way by those before him. The work is however systematic; and he seems to have been one of the first, after Boole,—with the exception of Peirce,—to realize what was wanted for a complete scheme of Symbolic Logic. Thus he introduces symbols for "the universe", and for "nothing"; and for the negation of a term or group of terms. Several of the distinctive characteristics of the non-exclusive notation are also to be found here; for instance the formulæ, 'If $a+b=b$ then $ab=a$', 'the negation of $a+u$ is $\bar{a}\bar{u}$', and so on. The general relation represented by the proposition is that of subsumption; and, like most others who adopt this interpretation, he uses a symbol which is intended to cover the two cases of coincidence and subordination of the classes. The particular sign adopted is $\angle$, one, namely, which is closely analogous to, though not quite identical with, the usual mathematical sign for inclusion in respect of magnitude.

13 and 14. These schemes are identical for all present purposes of enquiry. They are founded on a consistent application of the subsumption or inclusion view of the proposition, referring the subject class S to some part of the predicate class not-P. Peirce's notation was first proposed in 1870 (*American Journ. of Math.*, Vol. III.). The particular symbol employed is intended to indicate that this relation of subsumption covers the two actual class relations of coincidence and inclusion: it is a modification of the mathematical signs $=$ and $<$. Schröder's symbol is employed

in his later work (*Vorlesungen*); in his earlier work (*Operationskreis*) he had adopted the equational form. This too is constructed on the same analogy, the only difference being that he purposely avoids introducing the mathematical sign $<$ into his symbol. Both these authors make use also of the symbol of equality, in order to represent the case where the relation is reciprocal, i.e. where the classes are coincident: thus $S = \bar{P}$ (or the equivalent symbol) would represent the propositions 'All S is not-P' and 'All not-P is S'. Both authors also adopt the plan of representing the particular propositions as denials of universals. Thus they write respectively, $S \overline{=\!\!< P}$ and $S \not\in P_{\scriptscriptstyle|}$, for 'Some S is P'. The horizontal stroke (over copula and term) in the former case, and the vertical stroke (through the copula, and as suffix to the term) in the latter, representing negation.

15. This notation, adopted by Miss Ladd, now Mrs Ladd-Franklin, is founded on the conception of exclusion, either of one class from another or of one or more classes from the universe; and a symmetrical symbol is accordingly employed. It may be interpreted, 'S is excluded from P'. It is connected with the corresponding affirmative propositions by representing complete as contrasted with partial exclusion. Thus $S \vee P$ means that S is in part P, i.e. that it is *not* wholly excluded from P.

The following are some of the main characteristics of the scheme. Being symmetrical, the relation may be read off either way; as disconnecting S from P or P from S. The symbol $\overline{\vee}$ is the negation of $\vee$; so that the signification is particular when positive, and universal when negative. We may dispense with the sign 0, and refer everything to the universe (∞): thus $S \overline{\vee} \infty$, excluding S from the universe, declares that there is no S; and $S \vee \infty$ declares that there *is* S. The copula may be inserted at any point of a complex

expression; thus $ab\overline{\vee}cd = a\overline{\vee}bcd = abc\overline{\vee}d$: and, as 0 is dispensed with, the explicit reference to the universe may be omitted, and we may write simply $ab\overline{\vee}$ for, There is no ab.

The scheme is worked out in the Johns Hopkins *Studies*, and is applied to the neat and effective solution of some extremely complicated problems.

We now come to a group of forms which have been thrown together, as leading to an interpretation which is rather more in the way of implication than of subsumption: some of them indeed expressly describe themselves as indicative of implication.

16. The scheme of Darjes will be found in his *Weg zur Wahrheit* (1776). The same notation had however been already adopted in his *Introductio in Artem Inveniendi* (1747), towards the commencement of that work; but is scarcely, if at all, employed subsequently. His expression for the proposition in question is best put into words as, "posit S, and we sublate P". It turns almost entirely upon the representation of contradictories by + and −; a representation which, as in the closely analogous proposal of Segner already discussed, will do fairly well up to a certain point. If we only want to deal with pairs of contradictories, whether these consist of terms or of propositions, and only require to posit and sublate them, then the signs + and − are convenient. But then, as I have already pointed out, we lose the use of these signs for a far more appropriate purpose, viz. for that of logical aggregation and exception. Moreover the antithesis thus suggested of a contradictory, rather than a supplementary relation between S and not-S, soon leads to difficulties. How are we to represent not-S.not-P? By $(-S-P)$ or by $(-S)\times(-P)$?—there seems no convenient mode here for compounding terms and premises.

The most convenient rule perhaps for working this nota-

tion is found in its application to the process of logical conversion. From 'Posit S and sublate P', we deduce of course 'Posit P and sublate S'. Generalizing this to cover the four possible cases we see that the rule may be summed up in the words, Change the order of the terms and change both the signs: e.g. from $(+S+P)$ we infer $(-P-S)$:—but then we have to shake ourselves free from some rather strong mathematical associations in such a usage.

17. Mr McColl's scheme is another instance of results independently worked out, without knowledge of what had been already effected in the same way. But for this fact we might possibly not have had a new notation, as his symbolic method in itself seems to me to be practically identical with those of Peirce and Schröder. I have classed it amongst implications rather than subsumptions, because the author insists upon it as "a cardinal point of distinction" in his scheme that "every single letter, as well as every combination of letters, always denotes a *statement*." That such an interpretation can be consistently carried out seems certain, but, as I have tried to show in Chap. XVIII., the resolve to adhere to this will often involve an awkward circuit. If we are to have an exclusive logical interpretation I prefer the class interpretation.

Mr McColl adopts the sign (:) to denote implication, and a dash to denote negation; brackets being employed to group several terms together. Thus $x : (y+z)'$ would be read off, The statement x implies the contradiction of both y and z. The main convenience of this notation seems to me to consist in the ease with which we can thus express complex implications[1]. Thus $(x : y) : (z : w)$ stands for, 'The fact that x

[1] The earliest attempt at the expression of compound hypothetical expressions in this kind of way is, I believe, that of Maimon: "Hypothetische Sätze können durch das algebraische Verhältnisszeichen (:)

implies y implies that z implies w': x, y, z, w, being statements. Mr McColl has illustrated his method by a large number of examples in the *Educational Times* and elsewhere. His scheme is described in the *Proceedings of the London Mathematical Society*, Vols. IX., X.; and in *Mind*, No. XVI. The former account is mainly of a symbolic or mathematical character: the latter contains a good popular logical explanation of his views.

18. Pokorny's notation occurs in his *Neuer Grundriss der Logik* (1878). His scheme is of some interest, as having been worked out independently of those of McColl and Schröder; with some of whose results however it has much affinity, giving many of the equivalences of the former, and illustrating the Duality of the latter, though with a defective symbolic apparatus. He employs the sign $\frown$ for predication in universal propositions, and $\smile$ in particulars. Thus $a \frown$ not-b signifies 'All a is not-b'; and $a \smile$ not-b 'Some a is not-b'. As will be seen these expressions suffer from the great defect of adopting no simple sign for negative terms, or for the act of negation. The author appears to have no acquaintance with any analogous schemes except that of Drobisch.

19. Frege's scheme (*Begriffsschrift*, 1879) deserves to be called diagrammatic almost as much as symbolic. Here again we have an instance of an ingenious man working out a scheme,—in this case a very cumbrous one,—in apparent ignorance that anything better of the kind had ever been

angedeutet werden." Thus, with his employment of + to mark affirmation, the proposition 'If a is b then c is d' is represented by $a+b : c+d$ (*Versuch einer neuen Logik*, p. 69). But he does not work this conception out any further. The first really systematic attempt of this kind was probably that of R. Grassmann (*Begriffslehre*). The neat and effective rendering of a number of compound implications of this description is an excellent feature in Mr McColl's papers.

attempted before. A word or two only of explanation need be devoted to it here. A horizontal dash with a short vertical stroke at the end signifies a proposition: the line *S* running into *P* means that *P* is dependent upon *S*, this being in fact his symbol of dependency or implication. The short stroke under the *P*-line marks negation; so that the whole arrangement stands for 'If *S* then not *P*'. In this way we can proceed to build up more complicated dependencies. For instance, by joining this whole arrangement on to another such line, we can represent the compound dependency or implication, 'The fact that *S* implies the absence of *P* implies *Q*'; and so on. The obvious defect in this scheme is the inordinate amount of space demanded for its display. Nearly half a page is sometimes expended on an implication which, with any reasonable notation, could be compressed into a single line.

Nearly all the preceding schemes are those of logical innovators; some of them departing very widely from traditional arrangements, in the resolve to follow out the leading of their special symbolization and point of view. They mostly take, as the elements of comparison, *S* and not-*P*. The group next before us keeps for the most part more closely to tradition, taking *S* and *P* as the two elements whose relation has to be represented. Several of them in fact do little more than follow the details of the ordinary logic, merely disguising these under a mathematical dress. For want of any more suitable general designation I have classed them as belonging to the predicational group.

20. This is a form employed by Maimon (*Versuch einer neuen Logik*, 1794). I have found very great difficulty in understanding his notation, which indeed he seems to me to vary, and not always to use with perfect consistency. His scheme however deserves notice, for he evidently had a

strong conviction of the value of such notation; and, unlike some others who have ventured into symbols for little more than occasional purposes of illustration, he has systematically worked through the whole range of the traditional logic. The negative sign here indicates, as in so many other schemes, the contradictory of a class, so that $(-P)$ means not-P. The term x is intended to represent an arbitrary logical factor or determination. Hence the interpretation is, "S, howsoever determined, is not-P": i.e. by no kind of qualification can we reduce it to any part of P. Of course the qualification here can only be in the way of logical *determination*: not *abstraction*, as in some of the schemes to follow. There are a variety of serious defects in this notation, and it seems to me to be much less successful than that of some of his predecessors,—Lambert, for instance. The author has however contrived to work it through all the syllogistic moods. One obvious inaccuracy is to be noticed in the use of the sign $(=)$. We have no right to adopt the equational form unless the subject and predicate are identified, which is not the case with Maimon. He adheres, on the whole, to the customary predicational view.

21. This is the notation adopted by A. Victorin, in his *Neue natürlichere Darstellung der Logik* (1835):—an unusually elaborate symbolic treatment of the subject for the time at which it was written. By the symbol $(=)$ we ought, of course, to have a subdivision of (II.), by the identification of the subject-extent with a portion of that of the predicate. But this is quite foreign to the view actually adopted, the symbol being really one of simple predication. Accordingly we cannot directly change the order of the terms. From $S=P$ (the expression for 'All S is P') we cannot infer $P=S$, but only $\frac{1}{P}=S$:—it will be seen that we

have the same objectionable notation for particulars which has been already noticed on p. 87; in fact I rather think that Victorin was the inventor of it. The conversion of $S = -P$ gives, not $-P = S$, but $P = -S$. That is, the conversion has to be carried out entirely by logical rules, in defiance of mathematical usage. This seems a serious defect, and quite enough to destroy the value of the scheme.

The most original part of his scheme (for the time) is seen in his treatment of combinations of propositions. Thus from $A = +C$, $B = -C$; and $A = -D$, $B = +D$; we have respectively $A - B = C$, $A - B = -D$. These combined give $A - B = C - D$. A material example is given by combining 'Crime is punishable, and virtue is not: Crime is not praiseworthy, and virtue is'; into 'Crime, and not virtue, is punishable and not praiseworthy'. As the reader will recognize, the symbolic process is here scarcely more than an afterthought of the logical. The example closely resembles that of Mr Garden, discussed on p. 57, and is liable to the same errors in working, through the confusion between subtraction and negation.

22, 23. These two renderings employ purely arbitrary symbols, and are meant to do so, the symbols being merely substitutes for the copula of ordinary Logic. Wundt's symbol is one of a group (*Logik*, p. 244) some of which mark reciprocal relations between the terms, and some non-reciprocal, and its symmetrical form is intended to show that it belongs to the former class. Thus $S = P$ marks identity; $S > P$ superordination of S to P; $S < P$ subordination of S to P; and $S \between P$ mutual intersection of S and P. These, with $S\ )(\ P$, comprise the five distinct possible forms of objective class relation. To these however Wundt adds some others which are not so much actual class relations as dependencies or implications. De Morgan, I suspect, did not intend

thus to indicate the distinction between symmetrical and unsymmetrical forms of relation. His notation here is that which he adopted in his *Formal Logic:* he changed it subsequently in his papers in the *Camb. Phil. Transactions.*

24. The notation of Jäger was probably suggested by one of Ploucquet to be noticed next, from which it differs only by the sign of negation being prefixed to the predicate instead of to the subject. It does not play any important part in his treatise (*Handbuch der Logik*, 1839) though he frequently uses it to illustrate the rules of conversion and so forth. He is one of those who follow Victorin in the peculiar fractional representation of particular propositions; as do Lichtenfels (1842), Procházka (1842), and Kaulich (1869).

The remaining members of this group are distinctly indicative of the Quantification of the Predicate, the last two of them in particular being specially designed to call attention to this characteristic doctrine.

27, 28. The schemes of Bentham and of Hamilton may be passed over very briefly, as they play no part whatever amongst what can be called symbolic methods. They may both be translated here by the statement that the whole of S is distinct from the whole of P. Mr Bentham's t means totality, like Hamilton's (:), and the parallel lines of the one bear the same signification as the crossed wedge of the other, viz. 'distinct from', or as Hamilton sometimes puts it, 'not congruent with'. The differential characteristic of Hamilton's symbol lies in the distinction between the thick and thin ends of the wedge: according as the point is turned to the right or left it marks whether the proposition is to be interpreted in extension or in intension. This last is a subtlety in which he has scarcely found any one to follow him. For an account of the one scheme see Hamilton's *Logic*, II. p. 473; for that of the other see Bentham's *Logic*,

p. 134. The latter author, it need hardly be said, is best known as a distinguished botanist. There is a critical discussion of his claim to priority over Hamilton in the *Contemporary Review*, July, 1873.

26. Ploucquet has used more than one scheme in his various logical writings, but I believe they are all to be interpreted as quantifying the predicate. This one occurs in his *Fundamenta Philosophiæ speculativæ* (1759), and in his *Elementa Philosophiæ contemplativæ* (1778). It must be observed that the sign (–) here stands for affirmation, or rather for this and negation indifferently: the negation being affixed to the subject, where N stands for *nullum*. It is therefore merely a rendering of the common form, No S is P; whereas $S - P$ would have stood for, All S is P.

25. This is the form which Ploucquet seems mainly to employ in his more symbolic treatment of logic. It occurs for instance in his *Theoria Calculi Logici* (*Inst. Phil. Theoreticæ*, p. 47) and elsewhere. He there uses mainly two symbols: one for negation (>); and one for affirmation (juxtaposition of the letters). The former of these, it must be noticed, is quite arbitrary and somewhat misleading, for it has no suggestion, as usually elsewhere, of *inclusion*. The predicate, like the subject, is always quantified, the whole and part of it being indicated by capital and small letters respectively. Thus 'All A is B' stands, Ab, viz. 'All A is some B'. 'No A is C' stands $A > C$, viz. 'No A is any C'. The processes of reasoning are then mainly resolved into substitution of identities, and recognition of non-identities. It may be pointed out that had Ploucquet broken sufficiently with logical traditions to make a free use of negative predicates, or 'infinite' terms, he might have adopted another form for negative propositions. It is true that he does occasionally employ such predicates, but not sufficiently

often to make it worth while to keep a special symbol for them. Had he written, for instance, $\bar{P}$ for 'all not-P', and $\bar{p}$ for 'Some not-P', his expression for 'No S is P' would have been $S\bar{p}$, viz. 'All S is some not-P'. This would have been in closer accord with present symbolic usage, as exemplified in group (II.).

This notation of Ploucquet, (>), indicative of negation, arbitrary and misleading as it seems, has nevertheless found considerable acceptance. It is freely employed by H. C. W. Sigwart in the successive editions of his *Logik* (1818—35), as well as by others. Indeed E. A. von Schaden (1841) employs it when giving the "herkömmliche Ausdruck" of each figure. Somehow it seems also to have found its way very early into this country, in at least one instance. Thus in the anonymous *Logicæ Compendium* (Glasgow, 1764) I find "præterea est = signum propositionis affirmantis, > signum negantis" (p. 64); but there is no hint as to whence the sign is derived.

We now turn to a group the interpretation of which is distinctly and avowedly one of intension, that is, one in which the letters stand for notions or attributes instead of for classes or individuals.

29. Leibnitz's formula occurs in his *Non inelegans specimen demonstrandi in abstractis* (Erdmann, p. 94). It is not definitely assigned as a symbolic expression of the proposition; and like some other of his logical speculations in his shorter essays, seems to have been little more than a suggestion. Any hint however from such a quarter deserves attention; and there is a special reason in this case, owing to the more methodical proposals of Lambert and others, who undoubtedly took their impulse from Leibnitz.

His view appears to be this. The sign (–) stands here for 'detraction', or the removal of an attribute from a group

of attributes. This is not quite the same thing as ordinary logical 'abstraction', and the distinction is of some importance[1]. Starting with the conventional account, 'man' consists of 'rational animal'; i.e. we must superadd the attribute of rationality to those of animality in order to obtain humanity. Accordingly, if we abstract rationality we should be understood to leave simply animality behind; and since some animals (i.e. man) are rational, and some (i.e. brute) are irrational, this abstraction would not seem to lead obviously to a universal negative. But substitute for abstraction '*detraction*', i.e. entire removal of the attribute, and man without rationality is man no longer, i.e. is brute. We may express this by, Homo − rationalis = Brutum, which gives, 'No rational is brute'.

Leibnitz uses the sign (∞) for our (=). His words are:—"Aliud est detractio in notionibus, aliud negatio, v.g. Homo non rationalis est absurdum, seu impossibile. Sed licet dicere; simia est homo, nisi quod non est rationalis. Homo − rationalis aliud quam homo non rationalis. Nam homo − rationalis ∞ Brutum. Sed homo non rationalis est impossibile" (p. 96). "Sit $L - A \infty N$. Dico A et N nihil habere commune. Nam ex definitione detracti et residui, omnia quæ sunt in L manent in N, præter ea quæ sunt in A, quorum nihil manet in N" (p. 97).

30. This scheme proposed by Lambert might at first sight be considered identical with that of Holland (No. 9), or rather with the general propositional form of which that is a

[1] It is connected with the question of addition and subtraction in Logic, discussed on page 56. It may be remarked that Leibnitz had clearly realized the conditions under which subtraction is permissible. Thus he says (p. 97), "Sit $A \infty A$, dico reperiri posse duo B et N sic ut B non sit ∞N, et tamen $A+B \infty A+N$." Substituting our sign for equality, this is equivalent to saying that from $A+B=A+N$ we cannot infer, by subtraction of A from each side, that $B=N$.

particular case; for the two expressions are formally the same. In actual significance, however, they are in striking contrast with each other; Lambert's scheme having been offered in reply, as a sort of counter proposal to that of his friend. With Holland, the letters p and π, in the denominators, really stood for numerical factors. What he meant to signify was that 'some portion of the extent (estimated by $1 \div p$) of S is identical with some portion (similarly estimated by $1 \div \pi$) of that of P': though he failed when he came to interpret this into a negative proposition. But with Lambert, m and n have a different right to stand in the denominator. They mark *attributes*, and division by them stands for abstraction, as multiplication does for determination: the proposition being here interpreted not in respect of extent but of intent. His idea is this. Though S and P are distinct as classes they must have some attributes in common; that is, they must both belong to some higher genus. Abstract then certain attributes from each, as indicated in the division respectively by m and n, and the remaining groups of attributes will coincide.

This is quite true, and decidedly ingenious, but what one does not see is how this symbolic expression becomes a fitting representation of the universal negative rather than of any other proposition. Whatever the relations of extent of two different notions, S and P, it will always hold good that some of the attributes in one are different from some of those in the other. This points, I think, to an apparently essential defect in the attempt to interpret propositions in respect of the intent of both their subjects and predicates; it gives us, for instance, no means of distinguishing between, Some X is Y, and Some X is not-Y.

It is rather curious that Segner, whose work Lambert had read, could have set him right here. He has expressly

discussed almost exactly the same question, and realized its logical bearing clearly, though he did not reach the important symbolic step of introducing the inverse or division sign to mark it. He stated this theorem: Given that two classes, indicated by composite notions AB and CD, have something in common, and we abstract an attribute from each, say A and C, then the resultant classes, B and D, must also have something in common. But such community may be of four kinds, which he marks respectively $B=D$, $B<D$, $B>D$, $B\times D$; that is, coextension, inclusion of B in D, and of D in B, and intersection.

31. The last expression will be found described in Lambert's *Deutscher Gelehrter Briefwechsel*, I. 37; and in the *Nova Act. Erudit.* 1765. The present one is a slight modification of it given in his *Log. Abhandlungen*, I. 98. The general idea is exactly the same. Abstract sufficient attributes from P until only those are left which are common to it and to S. This does not yield an *identity* as before, for S is now more determinate than P, but it makes the remaining attributes of P *included*[1] in those of S. Interpreted therefore in intension, we have 'All $\frac{P}{m}$ is S', and this we express by the use of the sign $>$, in the form $S>\frac{P}{m}$. Another equivalent form given by Lambert, and which the reader will readily interpret, is $\frac{S}{m}<P$. It is obvious that in order to obtain an

[1] That Lambert regarded the inclusion as one of *intension* has been strangely overlooked by some logicians. Thus Bachmann (*Logik*, p. 148) says, "Ein Kopf wie Lambert konnte nicht einen allgemeinen bejahenden Urtheil durch $A>B$ bezeichnen,...da in keinem wahren allg. bej. Urtheile das Subjekt einen grösseren Umfang als das Predicat haben kann." He actually concluded that Lambert's MSS. must have been carelessly printed after his death, in order to account for this supposed error.

identity of subject and predicate, instead of a mere inclusion of one by the other, we must abstract from *both* of them, as in the preceding case.

32. This is the notation of Castillon, already noticed in Chap. XIX. The scheme deserves notice as being perhaps the only thoroughly consistent attempt at carrying out the intensive or notional interpretation. S is the subject concept; P the predicate, also regarded as a concept, and as forming part of the subject. M stands for the rest of the subject, after abstraction of P. The equation indicates the identity of the concept under these two aspects. In the case, as here, of a negative proposition we write $(-P)$. The formula therefore declares that the subject concept is composed of the negation of P, *plus* an indeterminate number of other attributes. It may be remarked that the use of the negative sign $(-)$ is very misleading, and has induced Castillon, like others who have used it, to treat it like the algebraical sign, and to transfer it as $(+)$ to the other side of the equation. It is plain that when we say 'No men are immortal', we are not subtracting from the notion *man*, but are regarding immortality as one of the constituent attributes: i.e. it is as much a part of the sum of those attributes as is any other non-essential quality.

33. The notation here is the same as that of Schröder, but the interpretation is quite distinct: in fact the author, Dr E. G. Husserl, introduces his scheme in opposition to the usual class rendering, of which he regards Schröder as the prominent exponent. I have given some account of the scheme in Chap. XIX., and need therefore only repeat here that to my thinking it occupies an intermediate place between the true intensive and extensive renderings. The symbols may be read off, 'an object of concept S is, as such, *not* an object of concept P'; or, so far as anything has the

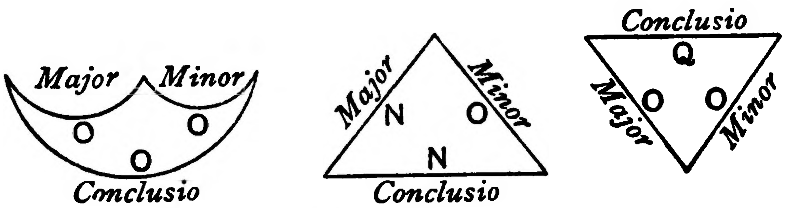
marks of S it has not those of P. The individual is supposed to be singled out by its marks,—i.e. by its intension, —and these are considered as signs of the concept, or marks, contained in the predicate. As the author points out, this interpretation of the proposition is very similar to that given by J. S. Mill. He has applied his interpretation in detail to the principal recognized formulæ of the Symbolic Logic. For the outline of his scheme see the *Gött. Gel. Anzeigen*, 1891: and the *Zeitschrift für wiss. Phil.*, 1891.

II. *On the employment of geometrical diagrams for the sensible representation of logical propositions.*

It is here proposed to take account of those graphical schemes only which deal directly with propositions, and which *analyse* them: that is, which exhibit, in some way or other, the subject and predicate separately, and indicate their mutual relation to each other. Hence several kinds of diagram, of great antiquity in Logic, can only be noticed and passed over. There is, for instance, the so-called Porphyrian Tree, which sometimes presents itself in a rather elaborate geometrical form. But this only represents the successive division of a genus into species, by dichotomy; and though of course giving rise to a number of essential propositions, it cannot be said in any way to represent them. Again there are the Squares to represent the Opposition of propositions, and similar ones to represent Opposition in the way of Modality; the former of which still hold their place in the text-books. Another kind, which at first glance might be taken for a complicated variety of one of these squares, is the Figure invented for helping the disputant

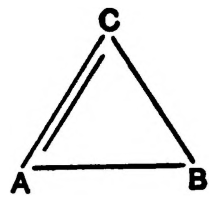
in the discovery of Middle Terms, and which was long known as the *Pons Asinorum.* A simple form of this will be found in Sanderson's *Compendium* (1680), and a decidedly more elaborate one in the Commentary on Porphyrius of Petrus Tartaretus (1581). They resemble the figure in Euclid's fifth proposition sufficiently to account for the transfer of name to the more familiar mathematical application.

Another variety of diagram, more nearly approximating to what we want, is to be found in certain figures which were long in use for representing the mutual relations of the terms in a syllogism. Their customary forms were these,—



They represent respectively the first three Figures; the fourth Figure, as well known, being commonly omitted. The letters *O*, *N*, *Q*, standing for *omne, nullum,* and *quoddam,* indicate the quantity of the proposition; so that we have here examples of *Barbara, Cesare,* and *Darapti.* The general conception involved seems plain, viz. that in the syllogism we connect the extremes *S* and *P*, by aid of a middle term,—as if going round by a triangle or along the two small semi-circles,—and thus get a relation which may be more simply displayed by the single path. These figures had a great vogue at one time, and in such works as the Commentaries of Alexander Aphrodisiensis, and Johannes Grammaticus, they occur by the hundred. An ample and learned account of their probable origin will be found in Hamilton's *Discussions*, p. 666. It is obvious that in dia-

grams of this description no kind of analysis of the proposition is attempted, and it cannot be claimed for them that they afford any real aid to the mind when dealing with trains of reasoning. For the last two or three centuries they have been entirely abandoned; the only exception here that I have met with being in the case of Reimarus, who in his *Vernunftlehre* has made use of them with a slight modification, employing double lines to mark affirmation and single lines to mark negation. Thus, with him, the figure

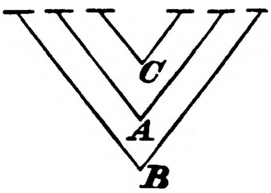


stands for a syllogism where A and C are connected affirmatively and C and B negatively; A and B being therefore connected negatively. The syllogism must consequently be *Cesare,* if we adhered to the old convention as to this position of the triangle being appropriate to the Second Figure. But no such modifications can give us any assistance in the analysis of propositions.

As regards then the employment of what we may call analytical diagrams,—that is, those which are meant to distinguish between subject and predicate, and between the different kinds of proposition,—there can be little doubt that their practical employment dates from Euler. That is to say, he first familiarized logicians with their use; and the particular kind of circular diagram which he employed has in consequence very commonly been named after him. But their actual origin is much earlier than this. In fact one would suppose that they must repeatedly have occurred to many logicians independently. Thus De Morgan states (and

I may repeat the remark in my own case) that he had himself hit upon Euler's scheme before he saw it anywhere represented. Indeed to any one accustomed to visualize geometrical figures it seems to me likely that Aristotle's *Dictum*, when once understood, would naturally present itself in the form of closed figures of some kind successively including each other.

Coming to historic facts, the earliest case in point which I have seen is in the *De Censura Veri* of Ludovicus Vives[1], where the mutual relation of the three terms in *Barbara*, as given by the two premises, is represented very much as on the Eulerian plan. He speaks of representing them by means of *triangles*, but the actual figures drawn are those of the letter V, as thus,



This is a mere isolated illustration, and is, I think, the only diagram to be found in the whole volume.

Priority in this direction has also been claimed by Hamilton for Alsted, who he maintains, had in the *Logicæ Systema harmonicum* (1614) anticipated the linear kind of diagram proposed by Lambert and which will presently be explained. I cannot however perceive that Alsted had the

[1] His words are: "Si aliqua pars *a* capit totum *b*, et aliqua pars *b* capit totum *c*, *c* totum capietur ab *a*: ut, si tres trianguli pingantur, quorum unus *B* sit maximus, et capiet alterum *A*, tertius sit minimus intra *A*, qui sit *C*, ita dicimus, si omne *b* est *a*, et omne *c* est *b*, omne *c* est *a*: adhibeatur regula quam diximus esse canonem artium et vitæ totius". (*De Censura Veri; Opera*, p. 607.) I do not understand how the capital and small letters here agree with each other.

slightest idea of representing what Euler and all subsequent logicians have aimed at representing. All that he says (p. 395; when speaking of the first Figure), is that the middle term is 'below' the major term, and 'above' the minor, and he just draws three lines of equal[1] length, one under the other, to illustrate what he means. "Etenim omne medium quod est inter duo extrema secundum altitudinem, id est inter extremum superius et inferius, illud inquam medium debet habere aliquod extremum supra se, et aliquod infra se. Atqui medius terminus in prima figura est talis. Habet enim terminum supra se, nempe praedicatum conclusionis in majore propositione positum, et habet terminum infra se positum, nempe subjectum conclusionis in minore propositione positum....Diagramma est tale:—

Animal
a ———————— *b*
sentit
c ———————— *d*
homo
e ———————— *f*."

There is nothing in this,—the only diagram of the sort which he gives,—even to suggest the distinction between affirmative and negative, universal and particular propositions; and this is surely the least which we can look for in these sensible illustrations. Nor can I find anything more to the point in any of his other logical works.

According to Hamilton (*Logic*, I. 256), Christian Weise, rector of the gymnasium at Zittau, was one of the first to make familiar use of logical diagrams. (But did Hamilton say this from his own observation? So far as I have been

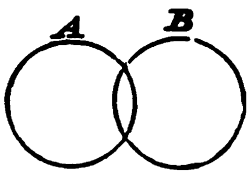
[1] This equality is of course significant. Hamilton, with that inaccuracy which seldom fails him where anything verging on mathematics is concerned, implies that the lines are of different length.

able to test his statements in this department of Logic, he is quite untrustworthy.) He says that he "finds" them in the *Nucleus Logicæ Weisianæ*, of which he calls Weise the author. But this was the work of J. C. Lange, published, as Hamilton says, in 1712, after Weise's death. I have not obtained a sight of the work, but suspect that the diagrams were contributed by Lange, who, as we shall presently see, was fond of this kind of thing. Lambert, who had evidently studied the work, says (*Architectonic*, I. 128) that it contains a number of illustrations of the syllogism by means of squares and circles. In Weise's own work, the *Doctrina Logica*,—the editions I had were published in 1690, and (almost unaltered) in 1711,—I cannot find the slightest suggestion of anything in the way of a diagram. Nor is it to be found in the *Curieuse Fragen über die Logica*, 1700.

In the only work by Lange to which I have been able to obtain access, viz. his *Inventum novum Quadratilogici*, there is nothing which strikes me as of any merit in this direction. There are a number of geometrical figures represented, both plane and solid, but the author does not seem to have grasped the essential conception of illustrating in this way the *mutual intersection*, or otherwise, of two or more classes. All that he represents is *continued subdivision*: e.g. that of A into B and C, of B into D and E, and C into F and G, and so on. All that this appropriately represents is the doctrine of Division, or continued Dichotomy. There is no attempt to represent the various relations of two terms, B and C, to each other, as involved in the various forms of proposition which have B and C for their subject and predicate.

We now come to Euler's well-known circles which were first described in his *Lettres à une Princesse d'Allemagne* (Letters 102 to 108:—the first of these is dated Feb. 14,

1761). That it was practically Euler who introduced these devices into Logic, there can be no doubt: in the sense that before his time they are never to be found in the ordinary manuals, and that since that time they have been more and more frequently introduced into such treatises. The weak point in this, and in all similar schemes, consists in the fact that they only illustrate in strictness the actual relation of classes to each other, rather than the imperfect knowledge of these relations which we may possess, or may wish to convey by means of the proposition. Accordingly, as I have shown in the first chapter, they will not appropriately fit in with the propositions of common logic, but rightfully demand the selection of a new group of suitable elementary propositions. This defect must have been noticed from the first in the case of the particular affirmative and negative, for the same diagram is commonly employed to stand for them both, which it does indifferently well:



$\left.\begin{array}{l}\text{Some } A \text{ is } B,\\ \text{Some } A \text{ is not } B,\end{array}\right\}$

for the real relation thus exhibited by the figure is of course "some (only) A is some (only) B", and this quantified proposition has no place in the ordinary scheme.

Euler himself indicated the distinction (at least so I judge from his diagram) by the position in which he put the letter A; if this stood in the 'A not-B' compartment it meant 'some A is not B', if in the AB compartment it meant 'some A is B'. But the common way of meeting the difficulty where it is at all recognized, is by the use of dotted lines to indicate our uncertainty as to where the boundary should lie. So far as I have been able to ascertain,

this plan (as applied to closed figures) was first employed by Thomson in the second edition of his *Laws of Thought* (1849)[1], but was doubtless suggested by the device of Lambert, to be presently explained. It has been praised for its ingenuity and success by De Morgan (*Formal Logic*, p. 323) and adopted by Jevons and a number of other logicians. Ueberweg has employed a somewhat more complicated scheme of a similar kind.

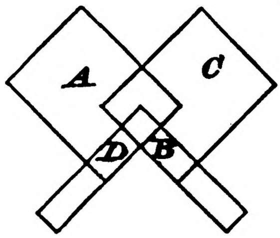
Any modifications of this sort seem to me (as already explained in Chap. I.) mis-aimed and ineffectual. If we want to represent our uncertainty about the correct employment of a diagram, the only consistent way is to draw *all* the figures which are covered by the assigned propositions and say frankly that we do not know which is the appropriate one. Of course this plan would be troublesome when several propositions have to be combined, as the consequent number of diagrams would be considerable. Thus in *Bokardo*, three diagrams would be needed for the major premise and two for the minor, making six in all.

The traditional logic has been so entirely confined to the simultaneous treatment of three terms only (this being the number demanded for the syllogism) that hardly any attempts have been made to represent diagrammatically the combinations of four terms and upwards. Almost the only serious attempt that I have seen in this way is by Bolzano. He was evidently trying under the right conception, for he was endeavouring to construct diagrams which should illus-

[1] In the first edition (1842) he noticed the ambiguity of the common representation, and substituted a circle for *B* and a half-moon entirely inside it for *A*. So far as this seems interpretable it must express "Some (only) *A* is some (only) *B*."

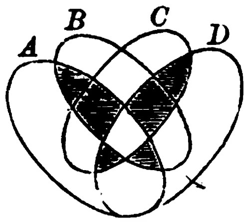
trate *all* the combinations producible by the class terms employed; but he adopted an impracticable method in using modified Eulerian diagrams. The consequence is that he has effected no general solution, though exhibiting a number of more or less ingenious figures to illustrate special cases. Thus a collection of distinct circles, included in a larger one, represents a number of species (mutually exclusive) comprehended under one genus; though, since the small circles cannot fill up all the contents of the large one we cannot thus conveniently represent the exhaustion of the genus by the aggregate of the species. A row of such circles, each of them interlinked with the next, represents the case of a succession of species each of which has something in common with the next, and so forth.

The most ingenious of his figures for four terms is the following. It is given here in order to show the necessary shortcomings of this method:—



This is offered in representation of the proposition "*A* which is *B* is the same as *C* which is *D*". Some trouble must have been taken to arrange it, so that as regards economy of time any such resort would be decidedly the reverse of an aid. Moreover, as the reader will readily perceive, it is not quite correct. One possible subdivision, viz. $\bar{A}B\bar{C}D$, has been omitted, for there is nothing in the statement forbid the occurrence of *BD* which is neither *A* nor Hence the correct state of things would be better

hibited thus, on the diagrammatic scheme adopted in this work :



With the exception of that of Bolzano, I have seen no attempt till quite lately to extend diagrammatic notation to the results of four terms, and it is only recently that really adequate figures have been proposed for those of three terms:—for instance both Drobisch and Schröder have used what we may call the three-circle diagram[1]. In saying this, I do not of course mean to imply that the problem was one of any particular difficulty, but merely state the fact that general satisfaction being felt with the Eulerian plan no serious attempts were made to improve upon it. Indeed, except on the part of those who wrote and thought under the influence of Boole, directly or indirectly, it was scarcely likely that need should be felt for any more generalized scheme.

The essential characteristic of the Eulerian plan being that of representing directly and immediately the inclusion and exclusion of classes, it is clear that the employment of circles as distinguished from any other closed figures is

[1] These writers merely represent in this way the class combinations or subdivisions as such: they do not adopt the subsequent step of using them as a basis for representing propositions. Dr J. Mich (*Grundriss der Logik*, 1871, p. 24) has come very near to this representation in one of his figures, where he employs three intersecting circles to represent class subdivisions, and then shades some of these. But though he uses these compartments to illustrate propositions, the shading itself has no propositional significance. The conception is still the Eulerian.

a mere accident. Nor have circles in fact always been employed. Thus Ploucquet,—whose system however, as he himself pointed out, in contrast with that of Lambert, is essentially symbolic and not diagrammatic,—has made use of squares. He claims quite correctly (*Sammlung*, p. 157) to have invented this method independently of, and in fact prior to, Euler. It is employed, for instance, in his *Fundamenta Philosophiæ speculativæ*, published in 1759: where *Barbara* is represented by three squares successively including each other. It is, I think, only employed twice, and the second application does not quite carry out the same principle. Kant (*Logik*, I. § 21) and De Morgan (*Formal Logic*, p. 9) have introduced or suggested both a square and a circle in the same diagram, one standing for subject and the other for predicate; with the view of distinguishing between these. Mr R. G. Latham (*Logic*, p. 88) and Mr Leechman (*Logic*, p. 66) have a square, circle, and triangle all in one figure, for the same purpose, presumably, of distinguishing between the three terms in the syllogism. Bolzano again, in one of the examples above adduced, has had resort to parallelograms: to which indeed, or to ellipses or to some such figure, it is evident he must have appealed if he wished to set suitably before us the outcome of four class terms. But to regard these as constituting distinct schemes of notation would be idle. They all do exactly the same thing, viz. they aim at so arranging two (or more) closed figures that these shall represent the mutual relation of inclusion and exclusion of the various classes denoted by the terms we employ.

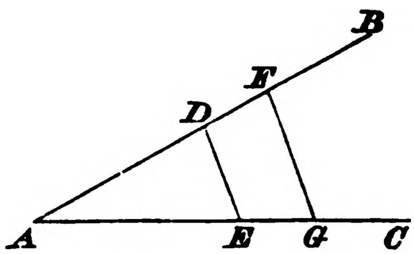
These Eulerian circles have certainly been the only form of diagram which can be said to have acquired any gene[ral] popularity. But it was some time before they found mu[ch] favour, or indeed were used for anything more than ve[ry]

occasional illustration. Perhaps the first logician to employ them freely, as a direct aid to the understanding in following the distinctions amongst the relations of propositions, was Krause. In his *Abriss des Systemes der Logik*,—my edition is of 1828, but he says that the first edition, of 1803, is practically the same in this respect,—he has worked out the problem which I have discussed on pp. 18—31, and with very similar results. It is plain that he had fully realized the relation in which these diagrams stand to the ordinary propositions, and the consequent necessity of displaying a plurality of diagrams for one syllogistic mood. He employs a peculiar device for representing the case of coincidence of extent; viz. that of drawing the two or three circles as if they had been slightly turned out of their common plane, round a common diameter, and were then seen in perspective.

There is one modification of this plan which deserves passing notice, both on its own account and because it has been so misjudged by Hamilton. It is that of Maass. In order to understand it we must recall one essential defect in the customary plan. Representing as this does the final outcome of the class relation, it is clear that any introduction of a fresh proposition into a problem may demand a diagram new from the beginning. If we have drawn a scheme for "All A is B", we must abandon it and draw another when "All B is A", instead of being able to incorporate the two into one. Seeing this, apparently, Maass took two fixed lines enclosing an angle, and regarded the third line which combined with them to form the necessary closed figure, as movable. Hence only one line had to be altered in order to meet the new information contained in such a second proposition. (*Logik*, p. 294.)

Thus let AB and AC be the fixed lines; and the triangle

ADE represent the class *X*, and *AFG* represent the class *Y*. If *FG* remain where it is we have "All *X* is *Y*", whilst in order to represent "All *X* is all *Y*" we have only to conceive *FG* transferred so as to coincide with *DE*. This



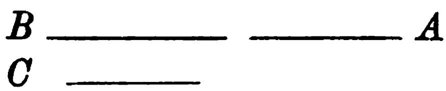
seems to me to constitute the essential characteristic of his scheme, which is worked out in a variety of figures of a more or less complicated kind. In some cases he starts with different angles of the triangle; but it is always areas, not angles, which are compared.

Hamilton who, as we know, never could succeed in grasping the nature of a triangle[1], entirely misconceived all this; and seeing that Maass began by talking of an angle, he concluded that angles were being employed as representative of class relations. Hence his judgment, hurled in a blast of wrathful and contemptuous condemnation, that this is "the only attempt made to illustrate Logic, not by the relations of geometrical quantities, but by the relations of geometrical relations,—angles" (*Logic*, II. 463).

The above schemes aim at representing the relative extent of class terms by the really analogous case of the relative extent of closed figures, and therefore tell their tale some-

[1] He went to his grave, apparently, with the belief that "every angle of every triangle infers, necessitates, contains, if you will, the whole of every other" (*Logic*, II. 463). As regards his judgement of Maass, I cannot but think that he was simply following Bachmann, who has made the same mistake.

what directly. A departure from this plan was made, very shortly after the date of Euler's letters[1], by Lambert, who introduced a more indirect scheme of diagrammatic notation. He indicates the extent of a class term by a straight line; the inclusion of one term in another being represented by drawing a shorter line under the other, the exclusion of two by each other by drawing them side by side, whilst the corresponding case to the intersecting circles is presented by drawing one line partially under the other, thus 
Hence *Celarent* might be represented:



No B is A,
All C is B,
$\therefore$ No C is A.

So far the scheme is of essentially the same kind as that of Euler, the only important difference being that the common part of the extent of the two terms is not here indicative of *identity;* for the line C is drawn under B and not made coincident with it. This was noticed at once by Ploucquet, whose theory of propositions turned entirely on the quantification of the predicate and consequent identity of subject and predicate. He criticizes the scheme at great length, maintaining amongst other objections, that Lambert would

[1] The analogy which Lambert actually had in view seems however to have been different. He evidently was influenced, like Alsted, by the technical expression, "thinking objects as *under* such and such a concept", which to modern ears would sound as little more than a play on words. Thus he draws a line to represent the general concept and puts a row of dots underneath to represent the individuals which stand under that concept. And again, "Ferner fordert der Ausdruck, dass alle A unter B gehören, von Wort zu Wort genommen, dass man die Linie A unter B setzen müsse" (*Dian.* § 181). The impropriety of this analogy was pointed out at the time by Holland (Lambert's *Briefwechsel*, I. 17).

do better to draw the second line, in an affirmative proposition, wholly or partially *coincident* with the first, and so secure this identity (*Sammlung*, p. 182). Since however the help to the eye would then be nearly lost, such an alteration would simply result in a poor and faulty imitation of the Eulerian scheme.

Lambert however did not stop here. Like most other clear thinkers he recognized the flaw in all these methods, viz. that we cannot represent the uncertain distribution of the predicate whilst we employ one and the same diagram for "All A is B", whether the predicate be 'all' or only 'some' B. He endeavoured to remedy this defect by the employment of dotted lines, thus:

B........———— ....
A ————

This means that A certainly covers a part of B, viz. the continuous part; and may cover the rest, viz. the dotted part, the dots representing our uncertainty. In this case the scheme answers fairly well, such use of dots not being open to the objection maintainable against it when circles are employed. But when he comes to extend this to particular propositions his use of dotted lines ceases to be consistent or even, to me, intelligible. One would have expected him to write 'some A is B' in some such way as this,

B ————.......
A ————.......

for by different filling in of the lines[1] we could cover the

[1] It may be remarked that such a diagram as this would be a very fair representation of the expression $x+\frac{0}{0}y=z+\frac{0}{0}w$ (see p. 301). We are equating, in each case, the sums of two elements; and the definite parts

case of there being '*B* which is not *A*', or *A* which is not *B*. But he does draw it

B ————
A ....

which might consistently be interpreted to cover the case of 'no *A* is *B*', as well as suggesting the possibility of there being no *A* at all.

Lambert's use however of this modification of this scheme is so obscure, and, when he comes to work out the syllogistic figures in detail, is so partially adhered to, that it scarcely seems worth the expenditure of further time. As a whole, the scheme seems to me distinctly inferior to that of Euler[1], and has in consequence been but little employed by other logicians. Many, no doubt, make a brief historic reference to it, or employ it for purposes of isolated illustration (*e.g.* Ulrich, H. C. W. Sigwart, and others); but none seem to have found in it any real aid in logical study. Hamilton offers a plan of somewhat the same sort (*Logic*, I. 189), but makes no subsequent use of it.

It may be remarked that Lambert's dotted lines were naturally objected to by Ploucquet, as being inconsistent with his own doctrine of Quantification. He modifies the

correspond to the continuous lines, and the indefinite to the dotted lines. All that we can positively assert in both cases is that the definite part of each figure cannot exceed the whole (definite and indefinite) of the other. We do not intend to identify the two definite parts, x and z.

[1] The above remarks apply to Lambert's scheme as he actually worked it; the purport of them being mainly historical. Dr Keynes has shown (*Formal Logic*, p. 243) that a slight modification in procedure will effect a great improvement, and, in his judgment, will render this plan less cumbrous, for syllogisms, than that of Euler. He displays *Datisi* thus,

P ————......
M ————
S ......————......

scheme accordingly, and has thus produced what is perhaps the only deliberate diagrammatic illustration of the doctrine in question. For instance he offers the following notation for the syllogism, 'All C is M: some B are not-M: therefore some B are not C' (*Sammlung*, p. 180):

$$\begin{array}{cc} & \underline{C} \\ \underline{\;b\;} & \underline{m} \end{array}$$

Here m represents 'Some M' *i.e.* the 'some' which is identified with 'All C': b represents the 'Some B' which is excluded from all M, and therefore from all the 'some M' marked by m. It is of course Lambert's diagram applied to the syllogism, 'All C is (all) m: no b is m: therefore no b is C.'

Another scheme has since been proposed by Mr Welton (*Manual of Logic*, p. 252) which resembles Lambert's, in so far as lines are employed, but partakes more of the nature of my own plan, so far as regards the interpretation. He displays 'All S is P' thus:—

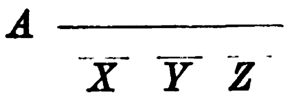
$$\begin{array}{cccc} \bar{S}\bar{P} & S\bar{P} & SP & \bar{S}P \\ |\cdots\cdots| & & |\text{———}| & \cdots\cdots| \end{array}$$

A space is provided for each possible alternative, and differently marked according to the positive and negative implications of the proposition. The blank space under $S\bar{P}$ means that this combination is excluded: the continuous line under SP means that this is secured: the dots under $\bar{S}P$ and $\bar{S}\bar{P}$ mean that no information is given about these. It will be observed that Mr Welton adopts here what may be called the conventional, as opposed to the symbolical interpretation, by considering that SP is definitely saved. In consequence his illustrations of I and O—at least as here represented—are the same respectively as those of A and E.

Hamilton's own system of notation is pretty well known. It is given in his *Logic* (end of Vol. II.) with a table, and is described in his *Discussions.* Some account of it will also be found in Thomson's *Laws of Thought.* It has been described (by himself) as "easy, simple, compendious, all-sufficient, consistent, manifest, precise, complete"; the corresponding antithetic adjectives being freely expended in the description of the schemes of those who had gone before him. To my thinking it does not deserve to rank as a diagrammatic scheme at all, though he does class it with the others as "geometric"; but is purely symbolical. What was aimed at in the methods above described was something that should explain itself at once, as in the circles of Euler, or need but a hint of explanation, as in the lines of Lambert. But there is clearly nothing in the two ends of a wedge to suggest subjects and predicates, or in a colon and comma to suggest distribution and non-distribution.

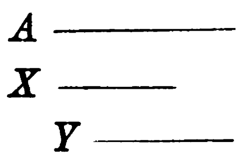
So far we have considered merely the case of categorical propositions; it still remains to say a few words as to the attempts made thus to represent other kinds of proposition. Hypotheticals may be dismissed at once, probably no logician having supposed that these should be exhibited in diagrams so as to come out in any way distinct from categoricals. Of course when we consider the hypothetical form as an optional rendering which only differs verbally from the categorical, we may regard our diagrams as representing either form indifferently. But this course, which I regard as the sound one, belongs essentially to the modern or class view of the import of propositions. Those who adopt the judgment interpretation can hardly in consistency come to any other conclusion than that hypotheticals are distinct from categoricals, and do not as such admit of diagrammatic representation.

The Disjunctive stands on a rather different footing, and some attention has been directed to its representation from the very first. Lambert, for instance, has represented what we must regard as a particular case of disjunction, viz. the subordination of a plurality of species to a genus, after this fashion :—



This of course indicates the fact that the three classes, X, Y, Z, together make up the extent of A.

It will be seen that we thus treat the species as mutually exclusive, and very appropriately so, such mutual exclusiveness being the natural characteristic of all true species. When however we attempt to adapt this linear scheme to the more comprehensive case of alternatives which are not mutually exclusive, we soon find that it fails us. *Two* non-exclusive alternatives indeed can be thus displayed, for such a scheme as the following will adequately mark the three cases



covered by "All A is either X or Y"; viz. that of any particular A being X only, Y only, and both X and Y. But make the same attempt with three classes, X, Y, Z; and we readily see that it breaks down. We cannot possibly represent, by lines, the seven cases covered by "All A is either X or Y or Z" (if the reader will try he will find that no arrangement will yield more than six of the needed combinations unless we make one of the lines discontinuous, by

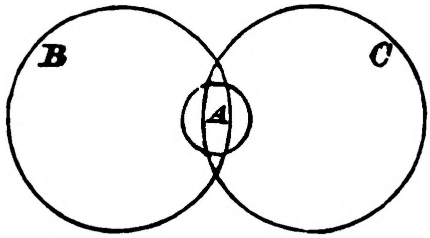
breaking it into portions), and accordingly we should be forced to appeal to closed figures which possess slightly greater capabilities in this respect.

Having just stated that Lambert has fairly represented a case of exclusive alternatives in disjunction, I must call attention to the fact that he expressly says that disjunctives do *not* admit of diagrammatic representation[1]. And his reason for so thinking deserves notice, as indicative of that deep distinction between the different accounts of the nature of propositions to which I have already had to allude. Starting with the assumption that *B* and *C* must be exclusive, he says that to represent '*A* is either *B* or *C*' we may begin by drawing the lines for *B* and *C* beside one another, but then comes in the uncertainty that we do not know under which of the two we are to set the *A* line. We have grounded on a mere hypothetical and can get no further. The explanation is to be sought in Lambert's point of view. What he does is to regard *A* as a *Begriff* or concept, in which case it becomes a unity, and we are then naturally in uncertainty as to whether to refer it to *B* or *C*. The case is much the same as if we had to exhibit the individual disjunctive 'Socrates is either awake or asleep'. But interpret *A* in its class extent, and the disjunctive '*A* is either *B* or *C*' becomes 'the classes *B* and *C* together make up *A*', which is essentially the same state of things as is readily represented by the subordination of species to a genus.

For the general representation of disjunctives, on this plan, Euler's circles do not answer better than Lambert's lines. In fact we cannot even represent 'All *A* is *B* or *C*

[1] "Die disjunctiven Sätze lassen sich gar nicht zeichnen, und zwar wiederum, weil sie nichts positives setzen." (*Neues Organon*, *Dian.* § 190.)

(only)' by circles, but are confined to 'All A is B or C (or both)', as thus:—



for if the B and C circles are not caused to intersect one another, the A circle will of course have to include something which lies outside them, and accordingly the conditions aimed at in the disjunction fail to be represented.

Kant (*Logik*, I. § 29) may be also noticed here as one of the very few logicians who have given a diagram to illustrate disjunctives. Like Lambert,—in fact like so many logicians,—he makes all disjunctives mutually exclusive. All he does indeed is to take a square and divide it up into four smaller squares; these four dividing members therefore just make up between them the whole sphere of the divided concept.

As will be gathered from the above, the attempts to give diagrammatic illustration of the disjunctive relation have been extremely few, and these few are mostly under arbitrary conditions. The same remarks might in fact be extended to any kind of symbolic representation of this particular relation, as the reader will have inferred from the notes in the preceding chapters (see pp. 54, 450). Almost the only attempt I have seen which deserves to be added to these is one by Hassler in his brief Syllabus of Logic (1832). He starts with the form $S \smile P$ to indicate an unconditional relation between subject and predicate. The hypothetical he writes $Sr \smile P$ to indicate that the relation is conditioned,

and that the ground of this condition lies in the subject; and the disjunctive is written $S \smile rP$, to indicate that the condition here lies in the predicate. In consequence such a doubly conditional proposition as, If an act is forbidden it must be by either law or command, is symbolized by $Sr \smile rP$. There is much in this which is open to criticism: amongst other things it either omits the case of disjunction in the subject, or simply classes it with hypotheticals.

Before concluding these historical notes I may briefly notice an application to which diagrammatic notation very readily lends itself, but which seems to me none the less an abusive employment of it. I refer to the attempt to represent quantitatively the relative extent of the terms. When, for instance, we have drawn, either by lines or circles, a figure to represent 'All *A* is *B*' it strikes us at once that we have got another element at our service; or, as a mathematician would say, there is still a disposable constant. We may draw the *B* circle, or line, of any size or length we please; why not then so draw it as to represent the relative extension of the *B* class as compared with the *A* class?

This idea seems to have occurred to logicians almost from the first, as was indeed natural, considering that the use of diagrams was of course borrowed from mathematics, and that a clear boundary line was not always drawn between the two sciences. Thus Lambert certainly seems to maintain that in strictness we must suppose each line to bear to any other the due proportionate length assigned by the extension of the terms. He even recognizes the difficulty in the case of a single line, viz. as to what length it should be drawn, resolving this however by the consideration that the unit of length being at our choice, any length will do if the unit be chosen accordingly. In the latter part of the *Neues Organon*,—where he is dealing with questions of Probability,

and the numerically, or rather proportionately, definite syllogism,—the length of the lines which represent the extent of the concepts becomes very important. So little was he prepared to regard the diagram as referring solely to the purely logical considerations of mere presence and absence of class characteristics, of inclusion and exclusion of classes by one another.

Of course if considerations of this quantitative kind were to be taken into account it would follow almost necessarily that circles should be abandoned in the formation of our diagrams; since their relative magnitude, or rather the relative magnitude of the figures produced by their intersection, is not at all an easy matter of intuitive observation. We should be reduced to the choice of lines or parallelograms, so that the almost exclusive employment of Eulerian circles has caused this quantitative application to be much less adopted than would otherwise probably have been the case. This application, however, has been made recently by F. A. Lange (*Logische Studien*), who in one of his Essays has made considerable use of diagrammatic methods in illustration of the Logic of Probability[1]. But I cannot regard the success of such a plan as encouraging. For the alternative forced upon us is this:—If we adhere to geometrical figures that are continuous, then the shapes of the various subdivisions soon become complicated; for, by the time we have reached six or even five terms, their combinations would result in yielding awkward compartments, whose

[1] Every student of Probability is of course familiar enough with the converse case, viz. that of reducing spatial relations to symbolic statement. Whenever we compute the chance that a ball dropped at random upon a frame-work will strike such and such a partition we are employing the same analogy as when we resort to diagrammatic representation of one of these quantitative logical propositions.

relative areas could not be estimated intuitively. If, on the other hand, we take our stand on having ultimate compartments whose relative magnitudes admit of ready computation we are driven to abandon continuous figures. Our *ABC* compartment, say, instead of being enclosed in a ring fence, is scattered about the field like an ill-arranged German principality of olden times, and its component portions require to be brought together in order to collect the whole before the eye. We draw a parallelogram to stand for *A*, and divide it into its *B* and not-*B* parts. If we divide each of these again into their *C* and not-*C* parts, we shall find before long that the corresponding compartments will not lie in juxtaposition with each other, and therefore the eye cannot conveniently gather them up into single groups. In fact such a plan almost necessarily leads to that primitive arrangement proposed by J. C. Lange, and mentioned at the outset of this discussion. Whatever elegance logical diagrams can possess, and whatever aid they can give to the mind through the sense of sight, seem thus to be forfeited.

My own conviction is very decided that all introduction of considerations such as these should be avoided as tending to confound the domains of Logic and Mathematics; of that which is, broadly speaking, qualitative, and that which is quantitative. The compartments yielded by our diagrams must be regarded solely in the light of being bounded by such and such contours, as lying inside or outside such and such lines. We must abstract entirely from all consideration of their relative magnitude, as we do of their actual shape, and trace no more connection between these facts and the logical extension of the terms which they represent than we do between this logical extension and the size and shape of the letter symbols, *A* and *B* and *C*.

INDEX.

INDEX OF BIBLIOGRAPHICAL REFERENCES.

It will be understood that this Index does not aim at being in any way exhaustive. It is merely a list of publications referred to as bearing on the topics specially characteristic of this work. It is confined also, with one exception, to works which I have been able to consult; and which seemed to me to be worth referring to. The asterisk marks those which are specially symbolic, in a logical sense, in the whole or some part of their treatment. For convenience of reference I indicate the date of the particular edition consulted. I may add that nearly all of these works are now to be found in the considerable logical collection in the Cambridge University Library (consisting of some 1200 volumes), of which a separate catalogue has been published. At the time when I commenced the serious study of Symbolic Logic many of the most important works which bore on the subject were not to be found in any of those great libraries in this country to which one naturally refers in the first place, and could therefore only be obtained by purchase from abroad. The best of the smaller libraries, for my purpose, was perhaps that of University College, London. I suppose that the almost entire abandonment of Logic as a serious academic study, for so many years in this country at least, had prevented the formation of those private professorial libraries, the frequent appearance of which in the market has kept the second-hand booksellers' shops in Germany so well supplied with works on this subject.

[1] Of these works by Ploucquet, the second is, as regards the logical part, a slightly altered edition of the first, with the addition of a *Theoria Calculi Logici*, in which the doctrine of the Quantification of the predicate is expounded. The third contains the same doctrine, with some slight novelties of notation. The fourth consists of extracts from the preceding and other works, with criticisms on them and replies by Ploucquet.

To the above list the following works may be added. I have found references to them of a kind which make me suppose that they must be of some interest on our subject; but every effort to discover them in this country, or to obtain them from abroad, has proved fruitless.

Busch, M. Anfangsgründe der logikalischen Algebra (Tübingen, 1768). Holland gathers, however (*Briefwechsel* I. 274), that it is "ein elendes Werk."

Schlosser, F. P. Disputatio de sororio Logices et Matheseos nexu, et applicatione præceptorum logicorum in disciplinis mathematicis (1727).

Tönnies, J. H. De Logicæ scientiæ ad exemplar Arithmeticæ instituenda ratione (1752):—referred to by Lambert (*Briefwechsel* I. 408) in a letter to Tönnies, as an "Inaugural Disputation"; and stated there to agree in general scope with his Grammatica Universalis (1771). It is briefly discussed by Lambert in a letter to Holland (I. 268).

CAMBRIDGE: PRINTED BY C. J. CLAY, M.A. & SONS, AT THE UNIVERSITY PRESS.